UNIVERSITY OF ARCHITECTURE, CIVIL ENGINEERING AND GEODESY
DEPARTMENT OF TECHNICAL MECHANICS

SELECTED TOPICS
ON
STRENGTH OF MATERIALS

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CHAPTER 1

SUBJECT OF THE STRENGTH OF MATERIALS. BASIC HYPOTHESIS

1.1. INTRODUCTION

Nowadays the building of structures, machines and other engineering structures is impossible without projects previously drawn. The project consists of the drawings and explanation notes presenting the dimensions of the construction elements, the materials necessary for their building and the technology for their building. The dimensions of the elements and details depend on the characteristics of the used materials and the external forces acting upon the structures and they have to be determined carefully during the design procedure.

The structure must be reliable as well as economical during the exploitation process. The reliability is guaranteed when the definite strength, stiffness, stability and durability are taken in mind in the structure. The economy of the construction depends on the material's expenditure, on the new technology introduction and on the cheaper materials application. It is obvious that the reliability and the economy are opposite requirements. Because of that, the Strength of Materials relies on the experience as well as the theory and is a science in development.

- **Basic concepts**

Strength is the ability of the structure to resist the influence of the external forces acting upon it.

Stiffness is the ability of the structure to resist the strains caused by the external forces acting upon it.

Stability is the property of the structure to keep its initial position of equilibrium.

Durability is the property of the structure to save its strength, stiffness and stability during the exploitation time.

Strength of Materials widely relies on the Theoretical Mechanics, Mathematics and Physics. Besides, it is the basis of the other subjects in the engineering practice.

1.2. BASIC PROBLEM OF THE STRENGTH OF MATERIALS

The basic problem of the science is *development of engineering methods to design the structure elements applying the restraining conditions about the strength, stiffness and stability of the structure when the definite durability as well as economy is given.*

1.3.REAL OBJECT AND CORRESPONDING COMPUTATIONAL SCHEME

To examine the real object a correct corresponding computational scheme must be chosen. The computational scheme is a real body for which the unessential attributes are eliminated. To choose the correct computational scheme the main hypotheses of Strength of materials have to be introduced.

1.4. MAIN HYPOTHESES

A. Hypotheses about the material building the body

- **Hypothesis of the material continuity**

The material is uniformly distributed in a whole body volume.

- **Hypothesis of the material homogeneity**

All points of the body have the same material properties.

- **Hypothesis of the material isotropy**

The material properties are the same in each direction of a body.

- **Hypothesis of the deformability of the body**

Contrary to the Theoretical Mechanics studying the rigid bodies, Strength of Materials studies the bodies possessing the ability *to deform*, i.e. the ability to change its initial shape and dimensions under the action of external forces.

The deformations at each point are assumed to be *small* relative to the dimensions of construction. Then, their influence onto the mutual positions of the loads can be neglected (the calculations will be made about the undeformed construction).

- **Hypothesis of the elasticity**

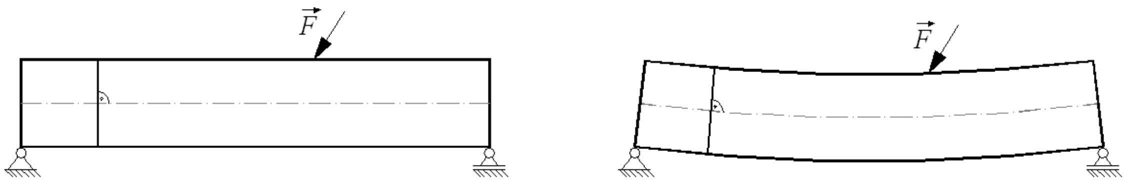
Elasticity is the ability of the body to restore its initial shape and dimensions when the acting forces have been removed.

B. Hypotheses about the shape of the body

- The basic problem of Strength of Materials is referred to the case of the *beam type* bodies. The beam is a body which length is significant bigger than the cross-sectional dimensions.

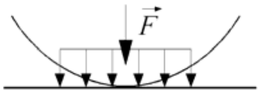
- **Hypothesis of the planar cross-sections (Bernoulli's hypothesis)**

Each planar cross-section normal to the axis of the beam before the deformation remains planar and normal to the same axis after deformation.



C. Hypotheses about the applied forces

- The distributed upon a small area loads are assumed to be concentrated.



- **Principle of Saint-Venant**

If we replace a set of forces acting upon an area Ω_1 of the deformable body with other set of forces equivalent to the first one, but acting upon the area Ω_2 of the same body, the replacing will influence on the stresses and deformations in the area Ω, containing Ω_1 and Ω_2, where the influence's magnitude will correspond to the size of the bigger area between Ω_1 and Ω_2.

- **Hypothesis of the local equilibrium**

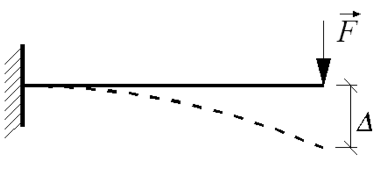
If the body is in equilibrium, then, each part of the body is also in equilibrium.

- **Hypothesis of the statical action of the forces**

The magnitude of the applied external forces increases gradually from zero to the final value.

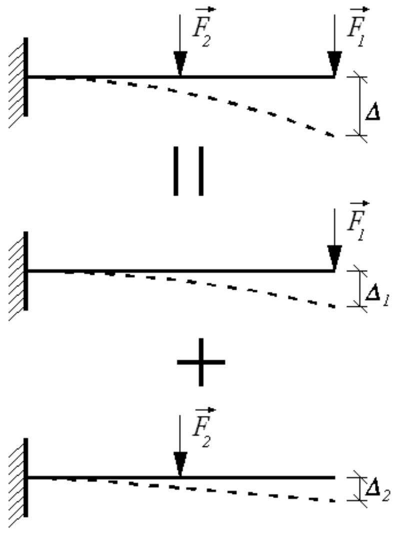
- **Hypothesis of the initial and final position of equilibrium**

Let the initial position of the beam to be the position of equilibrium. If the applied external forces cause the small deformations according to the hypothesis studied earlier, the final position of the beam is also position of equilibrium. Then, investigating the beam, the assumption that the initial position of equilibrium coincides with the final one is made.



- **Principle of superposition**

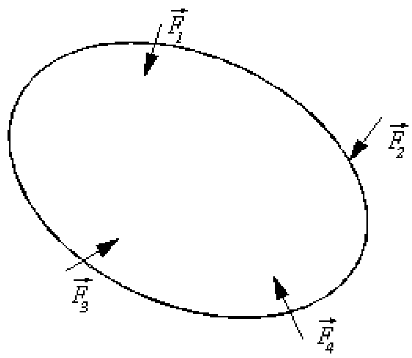
The final magnitude of a quantity considered (stress, strain, displacement, rotation) caused by the set of external forces can be obtained as an algebraic sum of the quantity magnitudes caused by the particular forces composing the set.



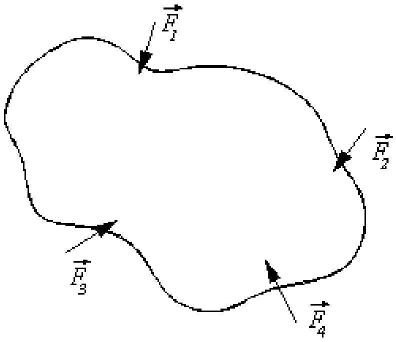
$$\Delta = \Delta_1 + \Delta_2$$

- **Principle of hardening**

A body has a definite shape and dimensions before loading.



The same body has the definite shape and dimensions after loading, again, but they are different than the first ones.



Rigid body – a body consisting of particles the distances between which do not change

Deformable body – a body consisting of particles the distances between which change. A deformable body is a rigid one only to the definite loading.

CHAPTER 2

INTERNAL FORCES

2.1. DEFINITION OF INTERNAL FORCES. METHOD OF SECTION.

A beam in equilibrium under the action of a set of forces is considered. This set of forces causes the deformation of the beam where the distances between the beam points change. Then, the forces of interaction between the points also change. The additional forces of interaction arising in the body are named *internal forces*. They have to be studied because they are related to the resistance of the body against the applied loads, and, consequently, to the strength of the body. The internal forces are the measure of interaction between two body parts situated on the two sides of the same section.

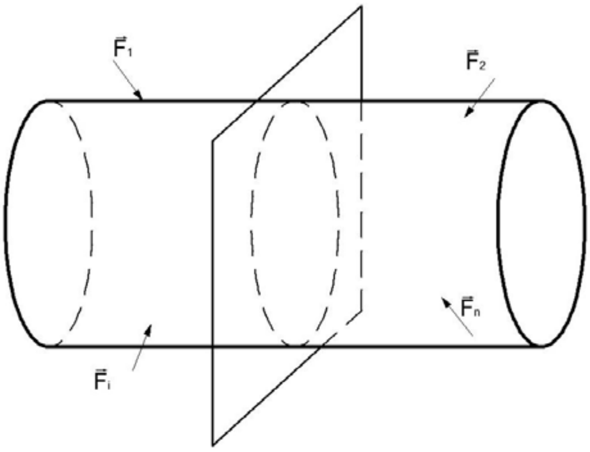


Fig. 2.1: A beam acted upon by a set of external forces

The internal forces can be determined by *the method of section*, as follow: Let the beam in fig.2.1 to be in equilibrium under the action of a set of forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, ...$\overrightarrow{F_n}$ named *external forces*. They include the external loads as well as the support reactions previously obtained. A plane normal to *the longitudinal axis of the beam* divides the body into two parts. A border section between these two parts is called *the cross-section*.

Further, one of the parts is removed (usually this one upon which the bigger number of loads acts) while the other will be investigated.

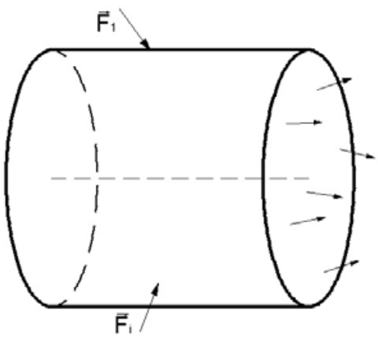


Fig. 2.2: The left beam part

The ***hypothesis of the local equilibrium*** has the essential role in Strength of materials and according to it, if a body is in equilibrium, then each part of the body is in equilibrium, too. This hypothesis leads to the conclusion that the left part of the beam must be in equilibrium under the action of a set of forces applied on it. However, the external forces are not in equilibrium themselves. To be the left beam part in equilibrium, the new type of forces must be introduced. These additional forces are *the internal forces* in the beam and they give the influence of the right beam part on the left one.

If the right beam part is chosen for investigation, then *the internal forces* giving the influence of the left beam part on the right one have to be put. According to the Newton's third law, the internal forces acting upon the left beam part and these ones acting upon the right beam part must have the same magnitudes, same directions and opposite senses.

The internal forces points of application in the plane of the cross-section are infinite as number and, because of that, they can not be found strictly. Then, to determine their magnitudes, the theorem of *Poinsot*[1] known by Theoretical mechanics will be used, as follow: reduction of the set of internal forces will be made about the cross-section's center of gravity where the main vector $\vec{R}$ and the main moment $\vec{M}$ will be obtained.

In a ***spatial case*** of loading when *the left beam part* is considered, the origin of the coordinate system is the center of gravity of the cross-section. The axis x is normal to the cross-sectional plane and its positive sense is out of the section. The axes y and z belong to the cross-sectional plane where the $z-$axis has the downward direction while the $y-$axis has the sense so that the three axes form the *right-handed* Cartesian coordinate system.

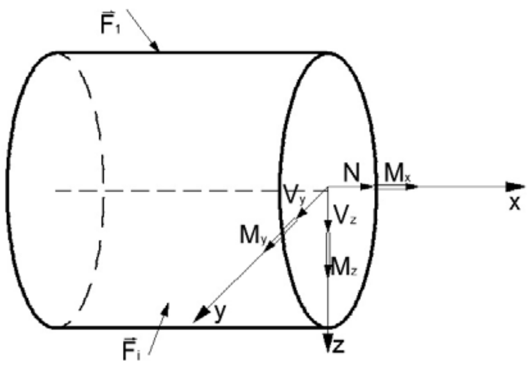


Fig. 2.3: Internal forces – spatial case

Vectors $\vec{R}$ and $\vec{M}$ are represented by its projections onto the axes of the right-handed Cartesian coordinate system. If the beam part considered is ***the left one***, then the senses of the internal forces always coincide with ***the senses of the axes***.

If the right beam part is chosen, then the $x-$axis of the right-handed coordinate system points ***toward the section***. Besides, all of the internal forces must be introduced with ***senses opposite*** to the senses of the axes.

The internal forces in the spatial case of loading are six and they are labeled in the following manner:

N **- axial (normal) force;**	T **- torsion moment;**
V_z **- shearing force onto** $z-$ **axis;**	M_y **- bending moment about** $y-$ **axis;**
V_y **- shearing force onto** $y-$ **axis;**	M_z **- bending moment about** $z-$ **axis.**

If the external forces acting upon the beam are situated in the plane containing the beam axis, then the loading case named ***planar*** is simpler: only the axes x and z have to be introduced in the cross-section's center of gravity. Now, the internal forces are ***three***:

N **- axial (normal) force;** V **- shearing force;** M **- bending moment.**

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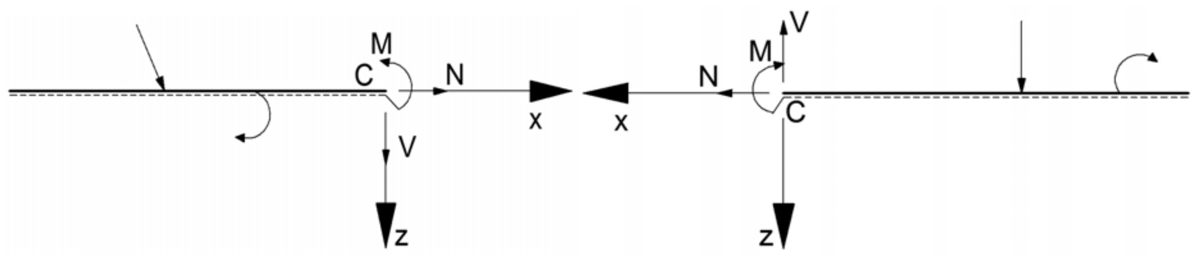
[1] *Louis Poinsot* (1777-1859) is a French mathematician.

The internal forces must be introduced *always* with their positive senses, which for the left and right beam part are given in fig.2.4:



Left beam part **Right beam part**

Fig. 2.4: Positive senses of internal forces – plane case

The axial force N is supposed to be positive when its sense is out of the section.

The shearing force V is supposed to be positive when its sense coincides with the sense of the positive axial force rotated at an angle of 90^0 in clockwise direction.

The bending moment is supposed to be positive when the curved arrow represented the moment begins from the downer beam end and finishes in the upper one without crossing the beam.

It is important to note, that the concept of internal forces always relates to the definite beam section.

2.2. INTERNAL FORCES FUNCTIONS AND DIAGRAMS

The conditions of equilibrium are written about the beam part considered. These equations are:

- In a ***spatial*** case

$$1)\ \sum_i F_{ix} = 0; \qquad 2)\ \sum_i F_{iy} = 0; \qquad 3)\ \sum_i F_{iz} = 0; \qquad (2.1)$$

$$4)\ \sum_i M_{ix} = 0; \qquad 5)\ \sum_i M_{iy} = 0; \qquad 6)\ \sum_i M_{iz} = 0; \qquad (2.2)$$

- In a ***plane*** case

$$1)\ \sum_i F_{ix} = 0; \qquad 2)\ \sum_i F_{iz} = 0; \qquad 3)\ \sum_i M_{i,C} = 0. \qquad (2.3)$$

It is obvious, that each internal force can be determined by one equation. However, in a real problem, it is not enough to find the magnitude of the internal forces in the definite beam section. It is necessary to obtain *the change of the internal forces* in the whole beam. To perform that, the beam must be separated into the segments.

The boundary point (section) of the segment is a beam point at which the concentrated force or moment is applied. If the distributed load acts upon a beam, then, both the beginning and the end of the load are the boundary points. Besides, the points at which the change of distributed load intensity exists are also boundary points. Finally, if the beam axis bends, then the bending point is a boundary point.

After that, an *arbitrary chosen* beam section of distance x for each segment must be considered. The distance x can be measured from the beginning of the beam, but in the most of the cases x is measured from the left or the right end of the segment. Further, the imaginary cut through the section chosen has to be made to divide the beam into two parts. Then, the one beam part has to be investigated and the equilibrium conditions must be written. In this manner, the internal forces will be obtained as functions of x.

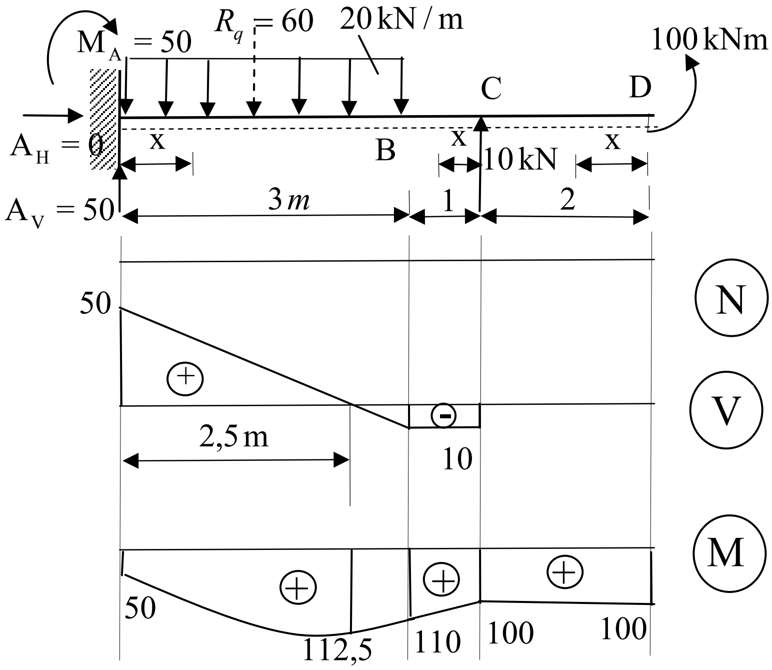
The graphs of these functions are named *the internal forces diagrams.* To build the diagrams, first *the zero line* representing the beam axis must be drawn in scale. The typical values of every function have to be drawn perpendicular to the zero line in a definite scale and the typical points have to be obtained. Finally, the points must be connected consequently.

The rules about the diagrams building are:

- *In a plane case of loading of a straight beam* – a broken line must be drawn under the beam axis. The positive values of the bending moment M must be put on the side of the broken line while the positive values of the axial force N and the shearing force V have to be put on the opposite to the broken line beam side;
- *In a plane case of loading of a bent beam* – The rule mentioned above is applied for each segment, but for a vertical or inclined segments the broken line represents a relatively named downer beam part;
- *In a spatial case of loading of a beam* – The values of N, V_z, T and M_y must be drawn parallel to $z-$axis. The positive values of N, V_z and T must correspond to the negative sense of $z-$axis while the positive values of M_y coincides with the positive sense of z. The values of V_y and M_z must be drawn parallel to $y-$axis where the positive values must be drawn on the side with negative sense of y.

The internal forces diagrams give the possibility to determine visually *the beam section at which the biggest internal force exists* (the failure of the construction starts at this beam section). Because of that, the internal forces diagrams predetermine the definite conditions of the construction strength, stiffness and reliability.

Problem 2.1. Build the internal forces diagrams of the planar straight beam given.



The support reactions are obtained by the equations:

$\sum_i F_{ix} = 0; \quad A_H = 0; \quad \sum_i F_{iz} = 0;$

$R_q = 20.3 - 60\ kN;$

$A_V - 60 + 10 = 0; \quad A_V = 50\ kN;$

$\sum M_A = 0; \quad -M_A - 60.1,5 +$

$+10.4 + 100 = 0;$

$M_A = 50\ kNm.$

Check: $\sum M_D = 0; \quad -M_A - 6A_V +$

$+60.4,5 - 10.2 + 100 = 0;$

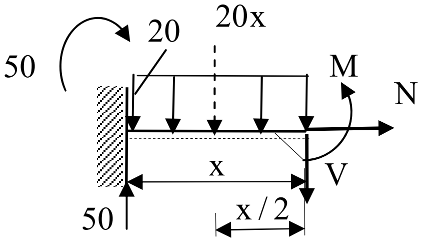
$-50 - 6.50 + 60.4,5 -$

$-10.2 + 100 = 0;$

$370 - 370 = 0.$

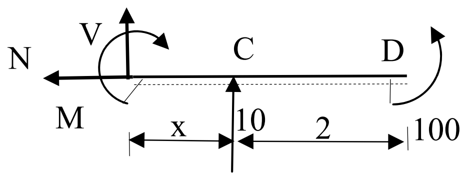
A beam given has three segments: AB**,** BC and CD and the internal forces functions are determined, as follow:

segment $AB: \quad 0 \le x \le 3m$



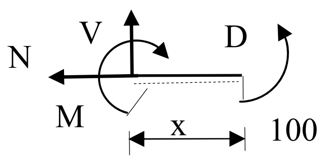
$$\sum_i F_{ix} = 0; \quad N = 0;$$
$$\sum_i F_{iz} = 0; \quad 50 - 20x - V = 0; \quad V = -20x + 50;$$
$$V(0) = 50kN; \quad V(3) = -50kN;$$
$$\sum_i M_{section} = 0; \quad M + 20x.\frac{x}{2} - 50x - 50 = 0;$$
$$M = -10x^2 + 50x + 50; \quad M(0) = 50kNm; \quad M(3) = -50kN;$$
$$V = -20x + 50 = 0; \; x = 2{,}5m; \; M(2{,}5) = 112{,}5kNm.$$

segment $BC: \quad 0 \le x \le 1m$



$$\sum_i F_{ix} = 0; \quad N = 0;$$
$$\sum_i F_{iz} = 0; \quad V - 10 = 0; V = -10 \; kN;$$
$$\sum M_{section} = 0; \quad -M + 10x + 100 = 0;$$
$$M = 10x + 100; \quad M(0) = 100kNm; M(1) = 110kNm.$$

segment $CD: \quad 0 \le x \le 2m$

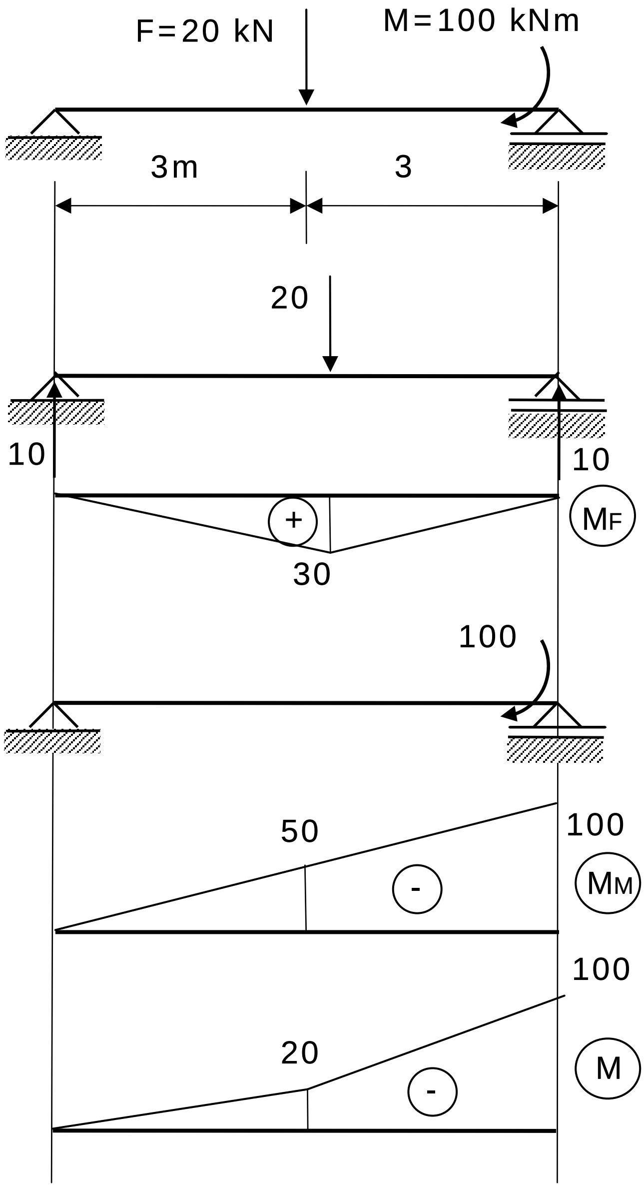


$$\sum_i F_{ix} = 0; \quad N = 0; \quad \sum_i F_{iz} = 0; \quad V = 0;$$
$$\sum M_{section} = 0; \quad -M + 100 = 0; \quad M =$$

To build the internal forces diagrams, *the principle of superposition* can be used, too. In accordance to the principle the final magnitude of a quantity considered (support reaction, internal force) caused by the set of forces can be obtained as an algebraic sum of the quantity magnitudes caused by the particular forces composing the set.

Problem 2.2 Apply the *principle of superposition* to build the bending moment diagram of the beam given.

The load applied on the beam consists of a concentrated force and a concentrated moment. First, the bending moment diagram under the action of a force will be built; then, the bending moment diagram under the action of a moment will be built. Finally, to obtain the entire bending moment diagram the typical values of the particular diagrams must be summarized.



2.3. THE DIFFERENTIAL EQUATIONS OF INTERNAL FORCES

2.3.1. IN THE PLANE CASE OF LOADING

The straight beam loaded by concentrated force, concentrated moment and *uniformly distributed transverse and axial loads* is considered (fig.2.5).

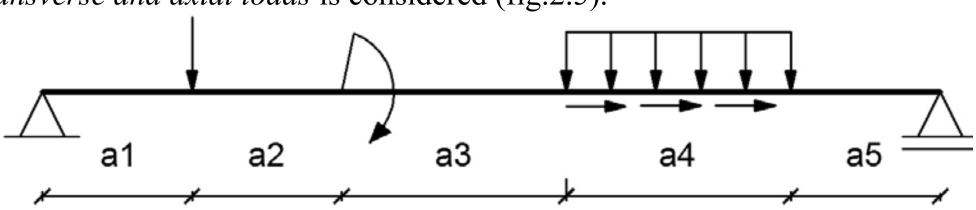


Fig. 2.5: Straight beam under loading

To derive the differential equations of the internal forces of the segment in which the distributed loads act the infinitesimal beam part is examined.

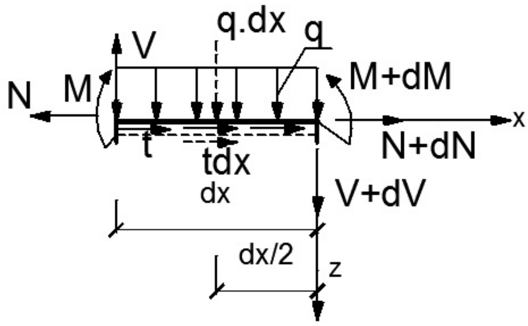


Fig.2.6: Infinitesimal beam part

Further, the equilibrium equations of the infinitesimal beam part are written:

$$1)\ \sum_i F_{ix} = 0; \qquad N + dN - N + tdx = 0; \qquad \frac{dN}{dx} = -t; \tag{2.4}$$

$$2)\ \sum_i F_{iz} = 0; \qquad V + dV - V + qdx = 0; \qquad \frac{dV}{dx} = -q; \tag{2.5}$$

$$3)\ \sum_i mom_{right}\overrightarrow{F_i} = 0; \qquad M + dM - M + qdx\frac{dx}{2} - V\,dx = 0. \tag{2.6}$$

The term $qdx\frac{dx}{2}$ is very small and it can be neglected. In this manner, the relation

$$\frac{dM}{dx} = V \quad \text{is obtained.} \tag{2.7}$$

It can be proved, if ***the distributed loads functions*** $t(x)$ ***and*** $q(x)$ ***are continuous functions***, then the differential equations of the internal forces are:

$$\frac{d\ N(x)}{dx} = -t\,(x);\ \frac{d\ V(x)}{dx} = -q\,(x);\ \frac{d\ M(x)}{dx} = V\,(x). \tag{2.8}$$

The distributed loads $t(x)$ and $q(x)$ are supposed to be positive when their senses coincide with the positive senses of the internal forces $N(x)$ and $\mathrm{V}\,(\mathrm{x})$, respectively, for the ***left*** beam part.

A small beam part of length Δx is considered. The distributed loads $q(\xi)$ and $t(\xi)$ represented as continuous functions act upon this small part of the beam.

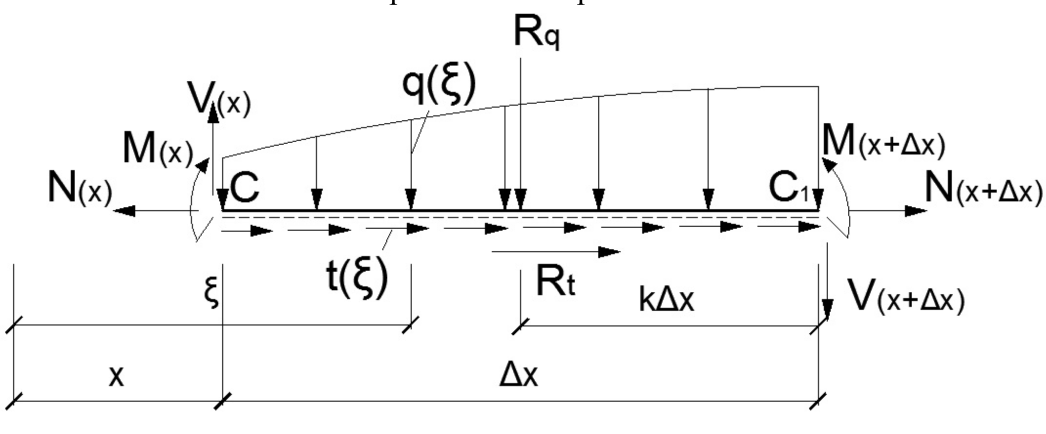


Fig. 2.7: A small beam part of length Δx

The coordinates x and ξ are measured from the left end of the segment. x is the distance to the left section's center of gravity of the beam part considered, while ξ is the distance to the arbitrary section of the beam part considered. The distributed loads have intensities $q(\xi)$ and $t(\xi)$, respectively, corresponding to distance ξ. The internal forces with their positive senses are introduced in the two sections of the part considered. Because of the small length Δx, the loads $q(\xi)$ and $t(\xi)$ can be considered as uniformly distributed with intensities equal to $q(\xi_1)$ and $t(\xi_2)$, where $x \le \xi_1 \le x + \Delta x$; $x \le \xi_2 \le x + \Delta x$. Then, the resultant forces are $R_q = q(\xi_1)\Delta x$ and $R_t = t(\xi_2)\Delta x$, and their point of application is on the beam part considered. The distance between the resultant forces point of application and the right beam end is $k\,\Delta x$, where $0 \le k \le 1$.

The beam part considered is in equilibrium and the equilibrium equations can be written, as follow:

$$1)\ \sum_i F_{ix} = 0; \qquad N(x+\Delta x) - N(x) + R_t = 0; \tag{2.9}$$

$$2)\ \sum_i F_{iz} = 0; \qquad V(x+\Delta x) - V(x) + R_q = 0; \tag{2.10}$$

$$3)\ \sum_i mom_{right}\overrightarrow{F_i} = 0; \qquad M(x+\Delta x) - M(x) - V(x)\Delta x + R_q k\Delta x = 0; \tag{2.11}$$

First, the expressions of R_q and R_t are substituted. After that, dividing by Δx, it is obtained:

$$\frac{N(x+\Delta x) - N(x)}{\Delta x} = -t(\xi_2); \frac{V(x+\Delta x) - V(x)}{\Delta x} = -q(\xi_1) \quad ; \tag{2.12}$$

$$\frac{M(x+\Delta x) - M(x)}{\Delta x} = V(x) - k\,q(\xi_1)\Delta x. \tag{2.13}$$

Further, the transition $\Delta x \to 0$ is made. In this case $\xi_1 \to x; \xi_2 \to x$. The equations (2.12) and (2.13) become equations (2.8).

These differential relations are correct when the distance x is measured from the *left* end of the segment. If it is measured from the *right* end of the segment, then the equations will have a form:

$$\frac{d\,N(x)}{dx} = t(x); \frac{d\,V(x)}{dx} = q(x); \frac{d\,M(x)}{dx} = -V(x). \tag{2.14}$$

2.3.2. IN THE SPATIAL CASE OF LOADING

The differential equations of the internal forces, when the distance x is measured from the *left* end of the segment, are:

$$\frac{d\,N(x)}{dx} = -t(x); \frac{d\,V_y(x)}{dx} = -q_y(x); \frac{d\,V_z(x)}{dx} = -q_z(x); \tag{2.15}$$

$$\frac{dT(x)}{dx} = -m_x(x); \frac{d\,M_y(x)}{dx} = V_z(x) \quad m_y(x); \frac{d\,M_z(x)}{dx} = -V_y(x) - m_z(x). \tag{2.16}$$

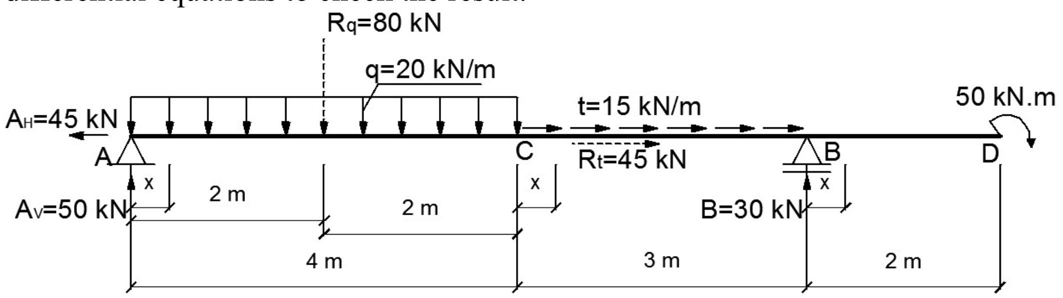
The functions of the distributed loads in the right side of these equations are: $t(x)$ - the intensity of the axial load; $q_y(x)$ and $q_z(x)$ - the intensities of the transverse loads parallel to the axes y and z, respectively; $m_x(x)$, $m_y(x)$ and $m_z(x)$ - the intensities of the distributed moments parallel to the axes of the Cartesian coordinate system.

If x is measured from the *right* segment end, then the differential equations will be:

$$\frac{d\,N(x)}{dx} = t(x); \frac{d\,V_y(x)}{dx} = q_y(x); \frac{d\,V_z(x)}{dx} = q_z(x); \tag{2.17}$$

$$\frac{dT(x)}{dx} = m_x(x);\ \frac{d\,M_y(x)}{dx} = -V_z(x) - m_y(x);\ \frac{d\,M_z(x)}{dx} = V_y(x) + m_z(x). \qquad (2.18)$$

Problem 2.3 Determine the internal forces functions of the beam shown and apply the differential equations to check the result.



First, the support reactions must be determined:

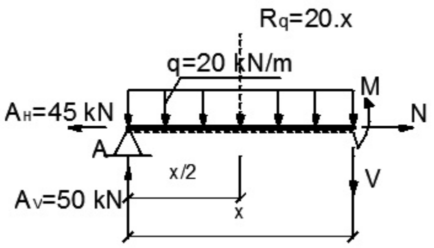
$\sum_i F_{ix} = 0; \quad 15.3 A_H = 0; \quad A_H = 45\ kN;$

$\sum M_A = 0; \quad 7B - 80.2 - 50 = 0; \quad 7B = 210; \quad B = 30\ kN;$

$\sum M_B = 0; \quad -7A_V + 80.5 - 50 = 0; \quad 7A_V = 350; \quad A_V = 50\ kN;$

Check:

$\sum_i F_{iz} = 0; \quad A_V + B - 80 = 0; \quad 50 + 30 - 80 = 0; \quad 80 - 80 = 0.$



segment $AC: \quad 0 \le x \le 4m$

$\sum_i F_{ix} = 0; \quad N - 45 = 0; \quad N = 45\ kN;$

$\sum_i F_{iz} = 0; \quad 50 - 20x - V = 0; \quad V = -20x + 50;$

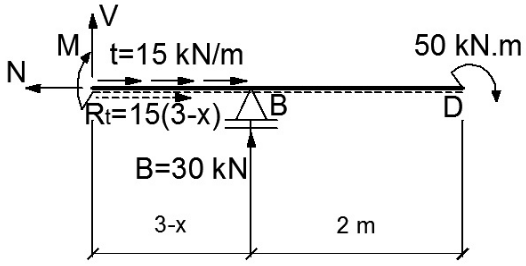
$\sum M_{section} = 0; \quad M + 20x.\frac{x}{2} - 50x = 0; \quad M = -10x^2 + 50x.$

Differential check (check by the differential equations of the internal forces): $t = 0; \quad q = 20\ kN/m;$

$\frac{d\,N(x)}{dx} = -t(x); \quad 0 = 0; \qquad \frac{d\,V(x)}{dx} = -q(x); \quad -20 = -20;$

$\frac{d\,M(x)}{dx} = V(x); \quad -20x + 50 = -20x + 50.$

segment $CB: \quad 0 \le x \le 3m$



$\sum_i F_{ix} = 0; \quad -N + 15(3 - x) = 0; \quad N = -15x + 45;$

$\sum_i F_{iz} = 0; \quad V + 30 = 0; \quad V = -30\ kN;$

$\sum M_{section} = 0; \quad -M + 30(3 - x) - 50 = 0;$

$M = -30x + 40.$

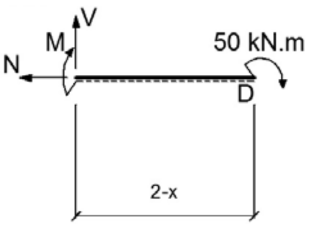
Differential check:

$t = 15\ kN/m;\quad q = 0;$

$$\frac{d\,N(x)}{dx} = -t\,(x);\quad -15 = -15; \qquad \frac{d\,V(x)}{dx} = \ q\,(x);\quad 0 = 0;$$

$$\frac{d\,M(x)}{dx} = V\,(x).;\quad -30 = -30.$$

segment $BD:\quad 0 \le x \le 2m$



$$\sum_i F_{ix} = 0;\quad N = 0;\quad \sum_i F_{iz} = 0;\quad V = 0;$$

$$\sum M_{section} = 0;\quad -M - 50 = 0;\quad M = -50.$$

Differential check:

$t = 0;\quad q = 0;$

$$\frac{d\,N(x)}{dx} = -t\,(x);\quad 0 = 0; \qquad \frac{d\,V(x)}{dx} = -q\,(x);\quad 0 = 0;$$

$$\frac{d\,M(x)}{dx} = V\,(x);\quad 0 = 0.$$

2.4. INTEGRATION OF THE INTERNAL FORCES DIFFERENTIAL EQUATIONS

This approach is applicable when a complicated distributed loads act upon a straight beam as well as a curved one. The essence of the method is the integration of the internal forces differential equations (2.8) in every beam segment.

To determine the integration constants *the boundary conditions* of equilibrium of typical beam sections must be written. These beam sections are separated by cuts at infinitesimal distance from the section. It is important *the unknown support reactions must not take part in the boundary conditions.*

After the internal forces functions have been determined, the internal forces diagrams can be drawn.

Problem 2.4 Apply the integration method to find the internal forces functions of the beam shown.

The beam contains two segments and the differential equations (2.8) are written and integrated for each of them, as follow:

segment $AC:\quad 0 \le x \le 4m$

$$t\,(x) = -5;\quad \frac{dN\,(x)}{dx} = -t\,(x);\quad \frac{dN\,(x)}{dx} = 5;\quad N\,(x) = 5x + C_1\,.$$

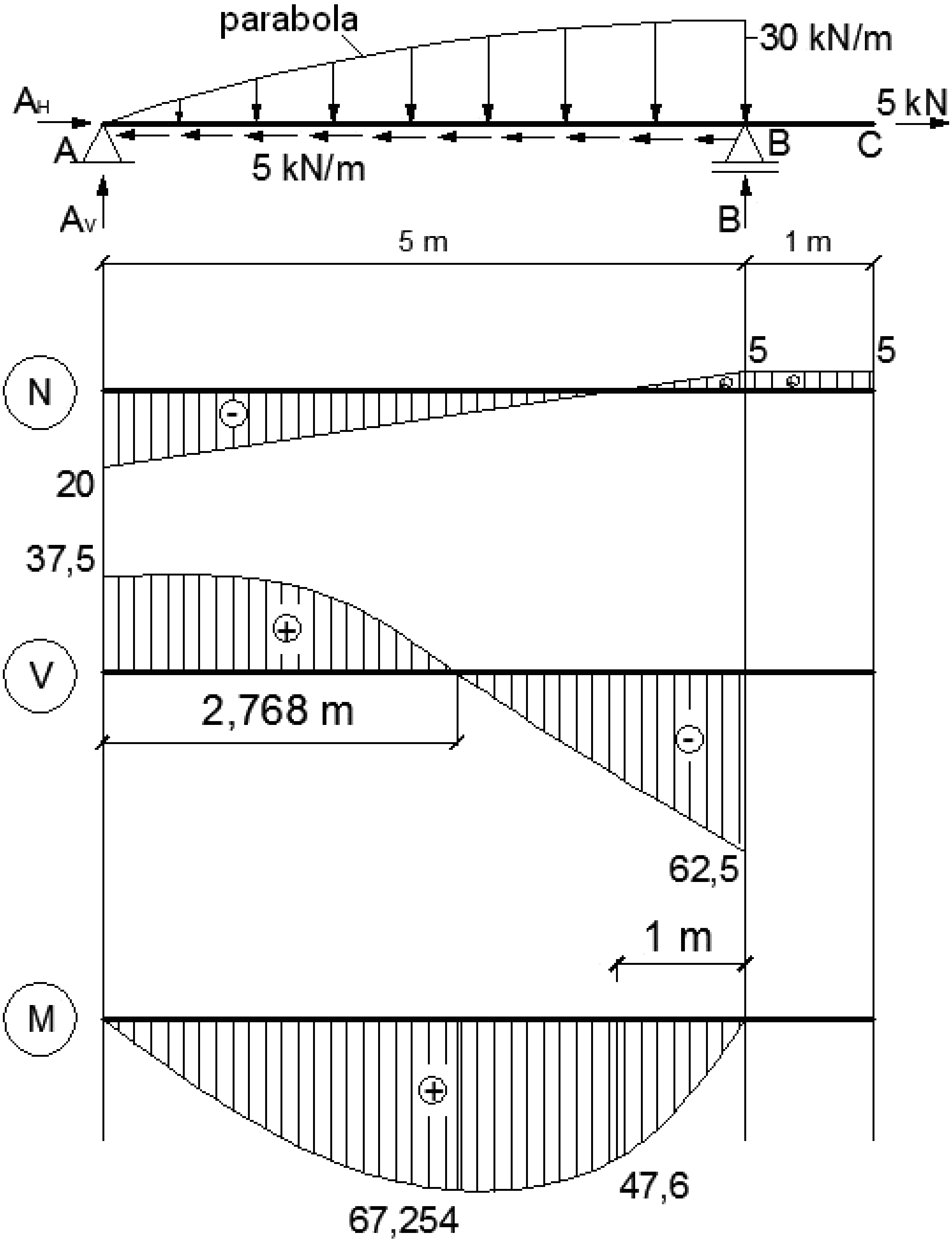
The function $q\,(x)$ is a parabola of a type $q\,(x) = ax^2 + bx + c$. To find constants a, b and c the conditions $q\,(0) = 0$; $q\,(5) = 30$ и $q'\,(5) = 0$ will be used. It is obtained: $a = -1{,}2$; $b = 12$; $c = 0$. Then, $q\,(x) = -1{,}2x^2 + 12x$.

The differential equation is written:

$$\frac{dV\,(x)}{dx} = -q\,(x);\quad \frac{dV\,(x)}{dx} = -\left(-1{,}2x^2 + 12x\right) = 1{,}2x^2 - 12x,$$

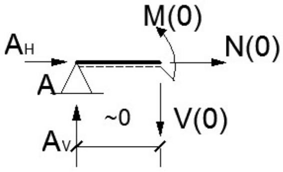
and after integration it is carried out: $V\,(x) = 0{,}4x^3 - 6x^2 + C_2$.

Further, the differential equation $\frac{dM(x)}{dx} = V(x)$; $\frac{dM(x)}{dx} = 0{,}4x^3 - 6x^2 + C_2$ is examined. The expression $M(x) = 0{,}1x^4 - 2x^3 + C_2 x + C_3$ is obtained after integration.



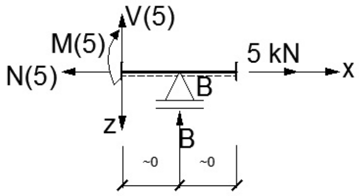
To find the integration constants the boundary sections *A* and *B* are investigated:

Boundary section A:



$\sum M_A = 0$; 1) $M(0) = 0$.

Boundary section B:



$\sum H = 0;$ 2) $N(5) = 5;$

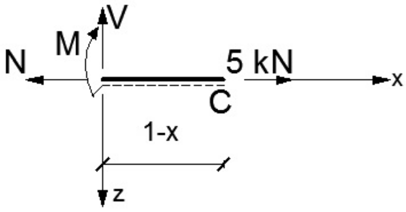
$\sum M_B = 0;$ 3) $M(5) = 0.$

The integration constants are: $C_1 = -20;\ C_2 = 37{,}5;\ C_3 = 0$.

The full expressions of the segment AB's internal forces functions have been already found and the diagrams can be built.

$$N(x) = 5x - 20;\quad V(x) = 0{,}4x^3 - 6x^2 + 37{,}5;\quad M(x) = 0{,}1x^4 - 2x^3 + 37{,}5x.$$

To find the internal forces functions in segment BC the method of section will be applied and the right segment part will be considered.



$\sum H = 0;\ N = 5;$

$\sum V = 0;\ V = 0;$

$\sum M = 0;\ M = 0.$

2.5. CHECKS OF THE INTERNAL FORCES FUNCTIONS AND DIAGRAMS

2.5.1. CHECK OF THE INTERNAL FORCES FUNCTIONS

The differential equations of the internal forces have to be considered:

- In a plane case of loading – equations (2.8);
- In a spatial case of loading – equations (2.12) and (2.13)

This check has been already performed in problem 2.3.

2.5.2. CHECK OF THE INTERNAL FORCES DIAGRAMS

a) Check about the type of the diagrams

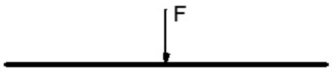
The last two differential equations in (2.8) are rearranged in a form

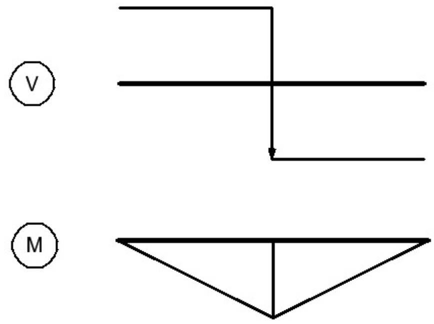
$$\frac{d^2M(x)}{dx^2} = \frac{d\,V(x)}{dx} = -q\,(x)\ . \qquad (2.19)$$

It is obvious, if in some beam segment $q(x) = const$, then the shearing force function $V(x)$ must be linear, while the bending moment function $M(x)$ must be square. If $q(x) = 0$, then $V(x)$ must be constant, while $M(x)$ must be linear function.

Furthermore, if in some beam segment the distributed load $q(x)$ points *down*, then the function $V(x)$ must *decrease*, and *the convexity* of $M-$diagram must *direct down*. However, if $q(x)$ points *up*, then the function $V(x)$ must *increase*, and *the convexity* of $M-$diagram must *direct up*.

b) Check about the steps and kinks in the diagrams

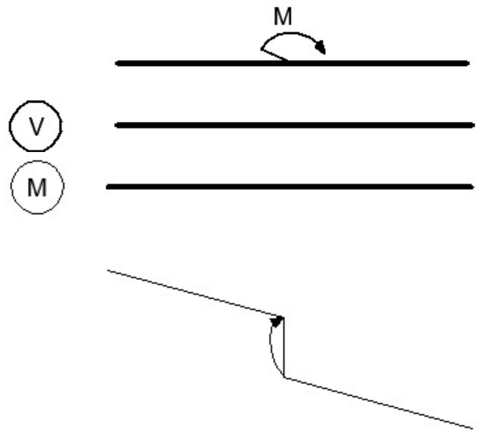




If a concentrated transverse force F is applied at some beam section, then the step in $V-$diagram must exist at the same section where the magnitude and the sense of the step coincide with these ones of the force. Besides, the kink in $M-$diagram must exist at the same beam section where the sense of the kink is in the sense of the force.

If a concentrated axial force F is applied at some beam section, then the step in $N-$diagram must exist at the same section where the magnitude of the step is equal to this one of the force while the sense of the step is the force's sense rotated at an angle of 90^0 clockwise.

If a concentrated moment is applied at some beam section, then the step in $M-$diagram must exist at the same section where the magnitude and the sense of the step coincide with these ones of the moment.



c) Area check

The differential equations (2.8) are considered for any segment and the rearrangements are made, as follow:

$$d\,M(x)=V\,(x)dx\,; \qquad \int_0^l dM(x)=\int_0^l V(x)dx\,, \tag{2.20}$$

where l is the length of the segment. The integral in the right side of the equation represents the area $A_{V,l}$ of $V-$ diagram. Then, using (2.20), it is obtained:

$$M(l)\text{-}M(0)=A_{V,l}\ . \tag{2.21}$$

The other two equations in (2.8) are integrated in the same manner:

$$d\,N(x)=\text{-}t\,(x)dx\,; \qquad \int_0^l dN(x)=\int_0^l -t(x)dx\,; \tag{2.22}$$

$$d\,V(x)=\text{-}q\,(x)dx\,; \qquad \int_0^l dV(x)=\int_0^l -q(x)dx\,. \tag{2.23}$$

Introducing $\qquad R_t=\int_0^l -t(x)dx$ and $R_q=\int_0^l -q(x)dx \qquad (2.24)$

which are the resultant forces of the distributed loads $t\,(x)$ and $q\,(x)$, respectively, the relations (2.22) and (2.23) become:

$$N(l) - N(0) = -R_t \ ; \ V(l) - V(0) = -R_q \ . \tag{2.25}$$

Problem 2.5. Make the area check for problem 2.1.

segment $AB: \quad 0 \le x \le 3m$

$$N(l) - N(0) = -R_t \ ; \ 0 - 0 = 0 \ ; \ 0 = 0 \ ;$$

$$V(l) - V(0) = -R_q \ ; \ -10 - 50 = -20.3 \ ; \ -60 = -60 \ ;$$

$$M(l) - M(0) = A_{V,l} \ ; \ 110 - 50 = (50 - 10).3/2 \ ; \ 60 = 60 \ .$$

segment $BC: \quad 0 \le x \le 1m$

$$N(l) - N(0) = -R_t \ ; \ 0 - 0 = 0 \ ; \ 0 = 0 \ ;$$

$$V(l) - V(0) = -R_q \ ; \ -10 + 10 = 0 \ ; \ 0 = 0 \ ;$$

$$M(l) - M(0) = A_{V,l} \ ; \ 100 - 110 = -10.1 \ ; \ -10 = -10 \ .$$

segment $CD: \quad 0 \le x \le 2m$

$$N(l) - N(0) = -R_t \ ; \ 0 - 0 = 0 \ ; \ 0 = 0 \ ;$$

$$V(l) - V(0) = -R_q \ ; \ 0 - 0 = 0 \ ; \ 0 = 0 \ ;$$

$$M(l) - M(0) = A_{V,l} \ ; \ 100 - 100 = 0 \ ; \ 0 = 0 \ .$$

d) Check about the equilibrium of a joint

First, the joint must be detached from the construction by the imaginary cuts through it. Then, if the concentrated force or moment acts at the joint, it must be put. Further, the internal forces with their correct senses have to be introduced in the cuts' sections and the equilibrium equations of the joint must be written. Finally, the equations obtained must be checked.

CHAPTER 3

STRESSES

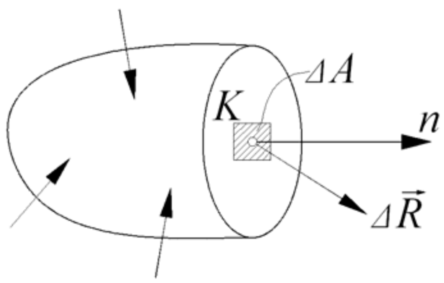
3.1. STRESS AT A POINT

A deformable body loaded by ***surface as well as body forces*** is in equilibrium. In order to investigate internal forces the body is divided into two portions by imaginary plane and the left portion is examined. Influence of the right body portion on the left one is accounted for the internal forces reduced about the cross-sectional center of gravity. Thus, the internal forces are defined as concentrated forces and moments.

Actually, the internal forces are distributed and their magnitudes are not constants in the cross-sectional area. Therefore, it is necessary to introduce the concept of ***stress*** which will characterize the law of distribution of internal forces.

A small area ΔA around point K in boundary plane of the left body portion is considered. The internal forces acting on the boundary plane give the influence of the right body portion on the left one. Some of the internal forces act on the small area ΔA only and they are reduced about point K. Statics proves that the result of the reduction of set of forces about point is a main vector and a main moment. However, the area upon which the forces act is very small and, thus, the main moment is neglected. Furthermore, it is supposed that the main vector $\Delta\vec{R}$ correctly describes the state of internal forces on the small area ΔA around the point.

The concept of ***stress*** was introduced by *Cauchy* in 1822. ***Stress*** is the intensity /density/ of the internal forces distribution on the small area around the point of the deformable body.



The ***average stress*** on the area can be described by the expression

$$\overrightarrow{p_{av}} = \frac{\overrightarrow{\Delta R}}{\Delta A}.$$

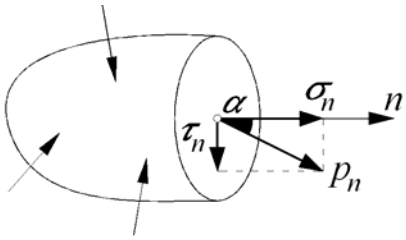
It is well-known that the material building the body is distributed uniformly in the whole volume of the body. Thus, the limit transition $\Delta A \to 0$ can be used. Then, ***stress on the area of normal n around point K is***

$$\overrightarrow{p_n} = \lim_{\Delta A \to 0} \frac{\Delta\vec{R}}{\Delta A} = \frac{d\vec{R}}{dA}.$$

The SI unit for stress is *Pascal* (symbol Pa) which is equivalent to one newton (force) per square meter (unit area): $1\ Pa = 1\ \frac{N}{m^2}$.

The vector of stress depends on the surface forces, body forces, on the position of the point considered, and on the orientation of the area around the point.

The stresses on the different planes passing through the point considered are different.



Generally, the stress vector is inclined at an angle with respect to the plane of the cross-section. Let α to be the angle between stress p_n and the cross-sectional normal ***n***. Then,

$\sigma_n = p_n \cos\alpha$ is the ***normal stress*** on the plane of normal ***n,***

$\tau_n = p_n \sin\alpha$ is the ***shearing stress*** on the plane of normal ***n***.

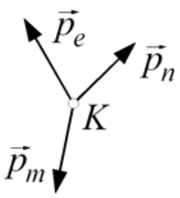
The normal stresses arise when the particles of the body strive either to remove or to approach each other. Shearing stresses are related to the mutual displacements of the particles in the cross-sectional plane.

It is evident that

$$p_n = \sqrt{\sigma_n^2 + \tau_n^2}\ .$$

The vectors of these stresses have the same origin. Then, their tails will lie on the ellipsoid of stresses, named Lame’s ellipsoid.

The state of stress at point K represents a sum of all stresses onto all possible planes passing through the point.

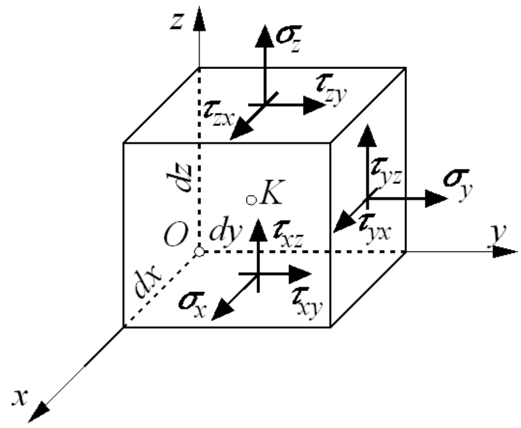


The investigation on the state of stress gives a possibility to analyze the strength of material when the random loading acts upon the body.

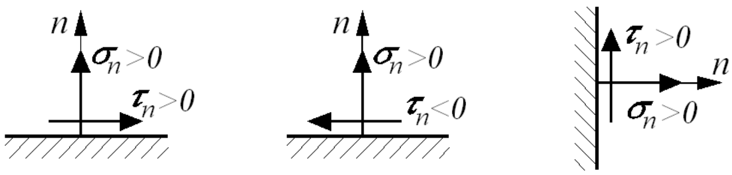
A body loaded by a set of external forces is given. An infinitesimal parallelepiped of dimensions ***dx, dy, dz*** in the vicinity of arbitrary chosen point of the body is separated. The normal and shearing stresses about the point investigated will act on the walls of the parallelepiped.

***Normal stresses* are written with *one index*. It corresponds to the letter of the coordinate axis parallel to the normal stress considered.**

Shearing stresses have two indices. ***The first one*** corresponds to the index of the normal stress of this wall while ***the second index*** is the letter of the coordinate axis parallel to the shearing stress considered.

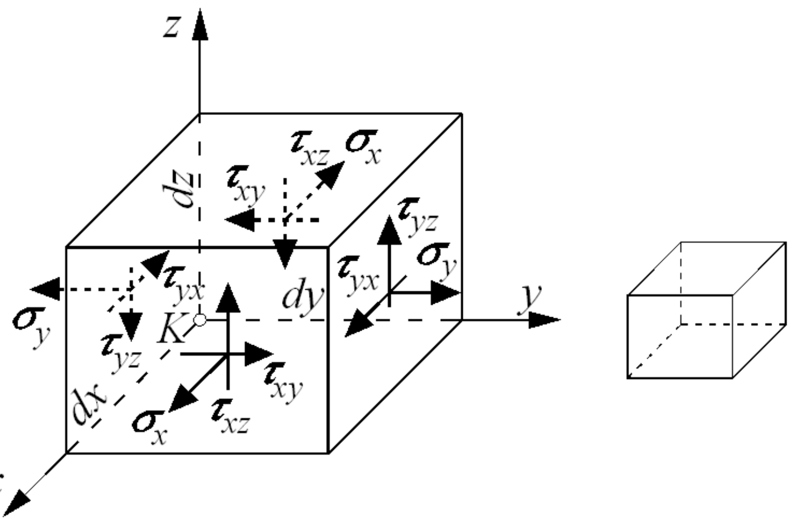


The behavior of the body acted upon by external forces does not depend on the coordinate system. Therefore, the state of stress can be described by ***tensor,*** named ***Cauchy's tensor.***



3.2. THEOREM OF THE SHEARING STRESSES EQUIVALENCE

The theorem gives the dependence between the magnitudes and directions of the shearing stresses acting on two mutually perpendicular planes around a point.



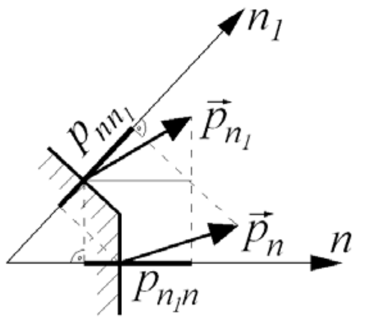
The moment equation of equilibrium about **z**-axis of the forces loading the walls of the parallelepiped is:

$$-\tau_{xy}dydzdx-\tau_{yx}dxdzdy=0\,;\tau_{yx}=\tau_{xy}$$

In the same fashion: $\tau_{yz}=\tau_{zy}$ and $\tau_{zx}=\tau_{xz}$.

Shearing stresses on two mutually perpendicular planes are equals. They are either "meeting" or "running" to each other.

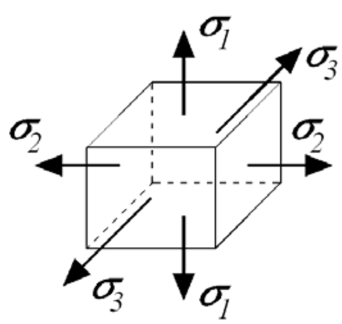
3.3. THEOREM OF THE TOTAL STRESSES EQUIVALENCE



The stresses on two planes of normal n_1 and n_2 passing through the same point of deformable body are given. **The essence of the theorem is that the projection of the first stress on the normal n_2 is equal to the projection of the second stress on the normal n_1.**

3.4. PRINCIPAL PLANES AND PRINCIPAL STRESSES

Augustin Louis Cauchy found that three mutually perpendicular planes onto which the shearing stresses take zero values exist at every point of the loaded body. These planes are named **principal planes,** their directions are named **principal directions,** and the normal stresses acting on these planes are named **principal stresses** labeled by $\sigma_1,\sigma_2,\sigma_3$.

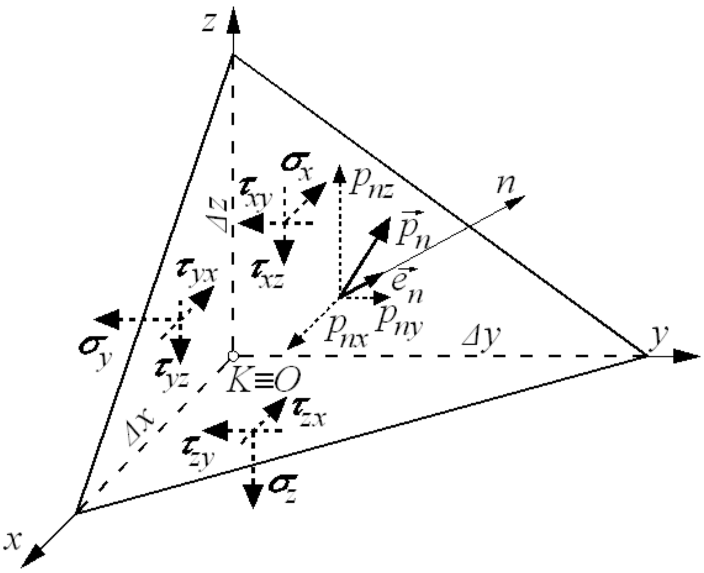


CHAPTER 4

THREE-DIMENSIONAL STATE OF STRESS AT A POINT

4.1. DETERMINATION OF THE STRESSES ON RANDOM PLANE AROUND A POINT WHEN THE STRESSES ON THREE MUTUALLY PERPENDICULAR PLANES ARE KNOWN

The state of stress at a point is known when the stress $\boldsymbol{p_n}$ on random plane around the point can be determined.



The stresses $\overrightarrow{p_x}\begin{matrix}\sigma_x\\ \tau_{xy}\\ \tau_{xz}\end{matrix}$; $\overrightarrow{p_y}\begin{matrix}\tau_{yx}\\ \sigma_y\\ \tau_{yz}\end{matrix}$; $\overrightarrow{p_z}\begin{matrix}\tau_{zx}\\ \tau_{zy}\\ \sigma_z\end{matrix}$ and the unit vector of the normal n

$\overrightarrow{e_n}\begin{matrix}\lambda = cos\,\alpha\\ \mu = cos\,\beta\\ \nu = cos\,\gamma\end{matrix}$ are given. The task is vector $\overrightarrow{p_n}\begin{matrix}p_{nx}\\ p_{ny}\\ p_{nz}\end{matrix}$ to be found.

$A_{\Delta ABC} = \Delta A;$
$A_{\Delta OBC} = \lambda \Delta A;$
$A_{\Delta OAC} = \mu \Delta A;$
$A_{\Delta OAB} = \nu \Delta A;$
$\Delta V = \frac{1}{3} \Delta A.h$

$$\sum_i X_i = 0;$$

$$\lim_{\Delta h \to 0} \left(-\sigma_x A_{\Delta OBC} - \tau_{yx} A_{\Delta OAC} - \tau_{zx} A_{\Delta AOB} + p_{nx} A_{\Delta ABC} + G_x \Delta V \right) = 0;$$

$$\lim_{h \to 0} \left(-\sigma_x \lambda \Delta A - \tau_{yx} \mu \Delta A - \tau_{zx} \nu \Delta A + p_{nx} \Delta A + G_x \Delta A.h/3 \right) = 0.$$

This expression is divided by h and, thus, p_{nx} is obtained. The expressions for p_{ny} and p_{nz} are obtained in the same manner.

$$\overrightarrow{p_n} \quad \begin{array}{l} p_{nx} = \lambda \sigma_x + \mu \tau_{yx} + \nu \tau_{zx} \\ p_{ny} = \lambda \tau_{xy} + \mu \sigma_y + \nu \tau_{zy} \\ p_{nz} = \lambda \tau_{xz} + \mu \tau_{yz} + \nu \sigma_z \end{array}$$

$$\overrightarrow{p_n} = \lambda \overrightarrow{p_x} + \mu \overrightarrow{p_y} + \nu \overrightarrow{p_z}$$

If the vectors of stresses on different planes passing through point K have the same origin, then the connection of their tails will form an ellipsoid named ***ellipsoid of the stresses (Lame's ellipsoid).***

4.2. STRESS TENSOR

The stress tensor is a sum of nine stresses and it is represented in the form

$$\begin{array}{ccc} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{array}.$$

Applying the theorem of the shearing stresses equivalence, namely $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, $\tau_{xz} = \tau_{zx}$, it can be concluded that only six stresses are independent of each other. These six parameters define the state of stress at point ***K***.

The normal stress on the plane of normal ***n*** is

$$\sigma_n = \sigma_x \lambda_n^2 + \sigma_y \mu_n^2 + \sigma_z \nu_n^2 + 2\tau_{xy}\lambda_n\mu_n + 2\tau_{xy}\mu_n\nu_n + 2\tau_{xy}\nu_n\lambda_n ,$$

while the shearing stress on the same plane will be obtained by expression $\tau_n^2 = p_n^2 - \sigma_n^2$.

4.3. PRINCIPAL STRESSES AND PRINCIPAL PLANES

There are three mutually perpendicular planes on which the shearing stresses take zero values. Normal stresses on these planes are named ***principal stresses*** and they can be obtained by cubic equation

$$\sigma_n^3 - a\sigma_n^2 + b\sigma_n - c = 0 ,$$

where $a = \sigma_x + \sigma_y + \sigma_z$;

$$b = \sigma_x \sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 =$$

$$= \begin{vmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{zy} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{xz} \\ \tau_{zx} & \sigma_x \end{vmatrix} ;$$

$$c = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{zx}^2 - \sigma_z \tau_{xy}^2 =$$

$$= \begin{vmatrix} \sigma_x & \tau_{yx} & \tau_{yx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yx} & \sigma_z \end{vmatrix} .$$

a, b and ***c*** are the three invariants of the state of stress at a point.

The roots of the cubic equation are always ***real*** and they are labeled by $\sigma_1, \sigma_2, \sigma_3$ $(\sigma_1 \geq \sigma_2 \geq \sigma_3)$. First root has the biggest value, i.e. it is ***maximum***, while the third root has the smallest value, i.e. it is ***minimum***, compared to all normal stresses existing on different planes passing through the point. The directions of $\sigma_1, \sigma_2, \sigma_3$, i.e. the normals n_1, n_2, n_3, of the planes of principal stresses are ***principal directions*** at the point. They are obtained by the set of equations

$$(\sigma_x - \sigma_n)\lambda_n + \tau_{yx}\mu_n + \tau_{zx}\nu_n = 0,$$
$$\tau_{xy}\lambda_n + (\sigma_y - \sigma_n)\mu_n + \tau_{zy}\nu_n = 0,$$
$$\tau_{xz}\lambda_n + \tau_{yz}\mu_n + (\sigma_z - \sigma_n)\nu_n = 0,$$
$$\lambda_n^2 + \mu_n^2 + \nu_n^2 = 1.$$

If the three principal stresses at a point are not equal to zero, then the state of stress at this point is called ***three-dimensional (spatial) state of stress.*** If two of the principal stresses are non-zero, then the state of stress is ***two-dimensional (plane).*** If only one of the principal stresses is non-zero, then the state of stress is ***one-dimensional (linear).***

4.4. EXTREME SHEARING STRESSES

The extreme values of the shearing stresses can be calculated by the formulas

$$\tau_1 = \frac{1}{2}(\sigma_2 - \sigma_3), \quad \tau_2 = \frac{1}{2}(\sigma_3 - \sigma_1), \quad \tau_3 = \frac{1}{2}(\sigma_1 - \sigma_2).$$

They are named ***extreme shearing stresses*** and they belong to the planes passing through the point and making the angle of 45^0 with the planes of principal stresses. The normal stresses on the planes of the extreme shearing stresses are

$$\sigma_{n,1} = \frac{1}{2}(\sigma_2 + \sigma_3), \quad \sigma_{n,2} = \frac{1}{2}(\sigma_3 + \sigma_1), \quad \sigma_{n,3} = \frac{1}{2}(\sigma_1 + \sigma_2).$$

CHAPTER 5

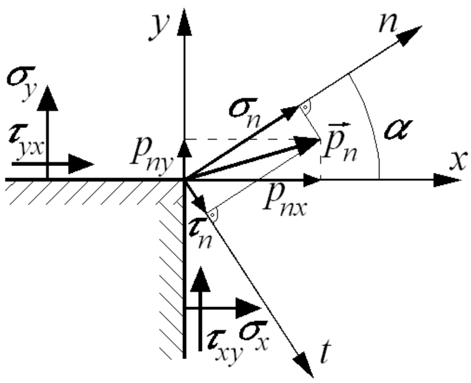
TWO-DIMENSIONAL STATE OF STRESS AT A POINT

5.1. DEFINITION

When one of the roots of the cubic equation mentioned earlier is equal to zero, then the state of stress is two-dimensional. Usually, $\tau_{zx} = \tau_{zy} = \sigma_z = 0$, i.e. the stresses different than zero lies in the plane ***xy***.

5.2. STRESSES ON A PLANE OF NORMAL N

The point ***K*** from the body loaded by external forces is considered. Stresses σ_x, σ_y and $\tau_{xy} = \tau_{yx}$ on the horizontal and vertical planes passing through the point are known. The task is to obtain the normal and shearing stresses on random plane of normal ***n*** making angle α with x-axis.



.

The projections of the total stress $\overrightarrow{p_n}$ acting on the plane considered are $p_{nx} = \lambda\sigma_x + \mu\tau_{yx}$ and $p_{ny} = \lambda\tau_{xy} + \mu\sigma_y$, where $\lambda = \cos\alpha$; $\mu = \sin\alpha$.

The normal and shearing stresses are expressed by these projections in the following manner $\sigma_n = p_{nx}\cos\alpha + p_{ny}\sin\alpha$; $\tau_n = p_{nx}\sin\alpha - p_{ny}\cos\alpha$.

Then, the expressions for p_{nx} and p_{ny} are substituted in these formulas. Further, by application of the shearing stresses equivalence theorem, namely $\tau_{xy} = \tau_{yx}$, the normal and shearing stresses on the plane are obtained as

$$\sigma_n = \sigma_x\cos^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha + \sigma_y\sin^2\alpha; \qquad \tau_n = (\sigma_x + \sigma_y)\sin\alpha\cos\alpha - \tau_{xy}(\cos^2\alpha - \sin^2\alpha)$$

After that, the trigonometric relations are used, as follow:

$\sin^2\alpha = (1-\cos 2\alpha)/2;\ \cos^2\alpha = (1+\cos 2\alpha)/2;\ \sin 2\alpha = 2\sin\alpha\cos\alpha;\ \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.

Thus, the normal and shearing stresses on the plane of a normal making angle α with horizontal axis are carried out:

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha; \qquad \tau_n = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\alpha - \tau_{xy}\cos 2\alpha$$

5.3. ANALYTICAL SOLUTION

a) Principal stresses and principal planes

In order to find the principal stresses condition $\frac{d\sigma_n}{d\alpha} = 0$ is used:

$$\frac{d\sigma_n}{d\alpha} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\alpha.2 + \tau_{xy}\cos 2\alpha.2 = 0 .$$

It is evident that the condition for the extremum of σ_n matches to the condition for the annulment of τ_n. Thus, it can be concluded that the shearing stresses are equal to zero ***on the planes of extreme normal stresses***. Further, the angles $\alpha_{1,2}$ of the principal directions are obtained by trigonometric equation

$$tg 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} .$$

The relation between two roots of this equation is $\alpha_2 = \alpha_1 \pm \frac{\pi}{2}$. It is obvious that the principal planes are perpendicular to each other.

The stresses invariants for the plane problem are $a = \sigma_x + \sigma_y;\ b = \sigma_x\sigma_y - \tau_{xy}^2; c = 0$.

The principal stresses are obtained using the following expression:

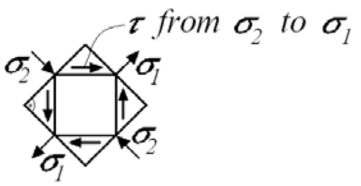
$$\sigma_{1,2} = \frac{1}{2}(\sigma_x - \sigma_y) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} .$$

b) Extreme values of shearing stresses

$$\frac{d\tau_n}{d\alpha} = 0;$$

$$tg 2\alpha = \frac{\sigma_x - \sigma_y}{2\tau_{xy}}; \quad \alpha_3 = \alpha_1 + \frac{\pi}{4}; \quad \alpha_4 = \alpha_3 \pm \frac{\pi}{2}.$$

$$\tau_{\max/\min} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}\ ;\ \sigma_{3,4} = \frac{1}{2}(\sigma_x + \sigma_y).$$



c) Planes of pure shear

$\sigma_n = 0$.

5.4. GRAPHICAL SOLUTION – MOHR'S CIRCLE

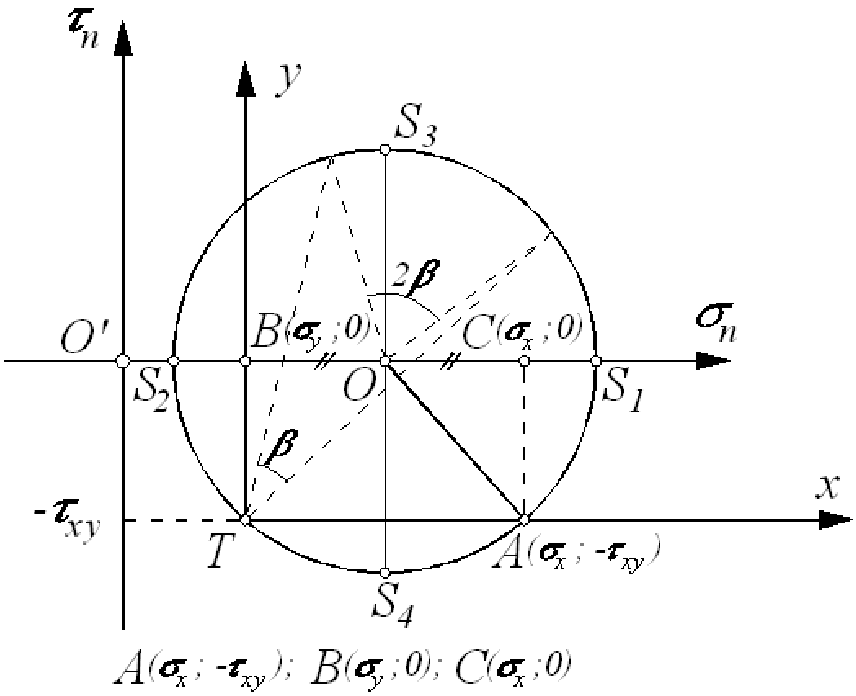
$$\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha;$$

$$\tau_n - 0 = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\alpha - \tau_{xy}\cos 2\alpha$$

$$\left[\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y)\right]^2 + (\tau_n - 0)^2 = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}{}^2;$$

$(x-a)^2 + (y-b)^2 = R^2$ - Equation of a circle

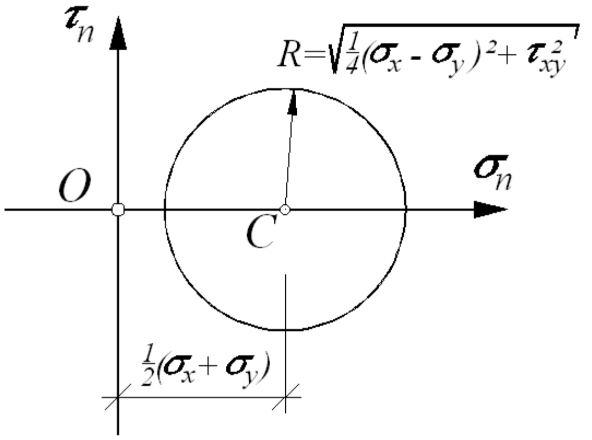
$$\left[\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y)\right]^2 + (\tau_n - 0)^2 = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}{}^2;$$



Center $C\left[\frac{1}{2}(\sigma_x + \sigma_y);0\right]$

Radius

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}\ ;$$



CHAPTER 6

STRAINS

6.1. BASIC NOTATIONS

Each loaded body is deformed. Its form and sizes are changed because the points of the body change their position.

Let a point $M(x;y;z)$ is an arbitrary point of loaded body. This point will take a position $M'(x';y';z')$ after the body's deformation.

The vector of the displacement $\vec{D}(u;v;w)=\overrightarrow{MM'}$ is defined. Its projections are

$$D_x=u(x;y;z);\;\; D_y=v(x;y;z);\;\; D_z=w(x;y;z). \tag{6.1}$$

$u(x;y;z)$, $v(x;y;z)$ and $w(x;y;z)$ are displacements along the axes x, y and z respectively.

Three points of the loaded body are shown

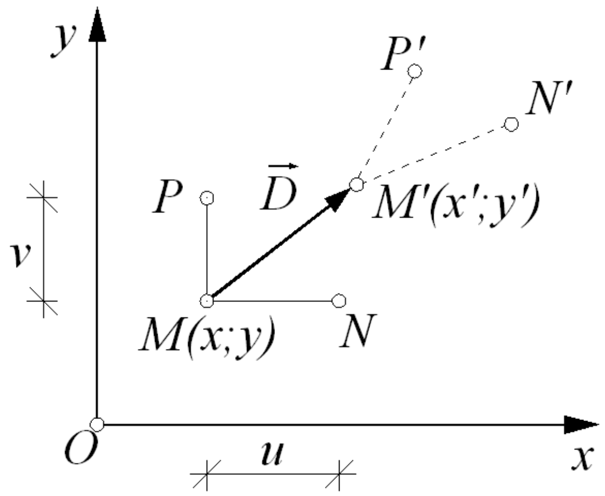


Fig. 6.1: Three points before and after the deformation

Linear deformation in a point M at the direction of the axis x is

$$\varepsilon_x=\lim_{MN\to 0}\frac{\overline{M'N'}-\overline{MN}}{\overline{MN}}.$$

Angular deformation γ_{xy} in a point M of the plane xy is the small angle of the change of the right angle between two directions before the body's deformation.

Deformed state is the combination of the linear and of the angular deformations on the axes and planes passing through this point.

6.2. DIFFERENTIAL EQUATIONS OF THE GEOMETRY (CAUCHY’S EQUATIONS)

Elementary parallelepiped is considered. It is connected with a coordinate system. Points $K(0;0;0)$and $A(dx;0;0)$ are chosen. Their positions after the deformation of the body are $K'(u;v;w)$ and $A'(dx+u+du;v+dv;w+dw)$.

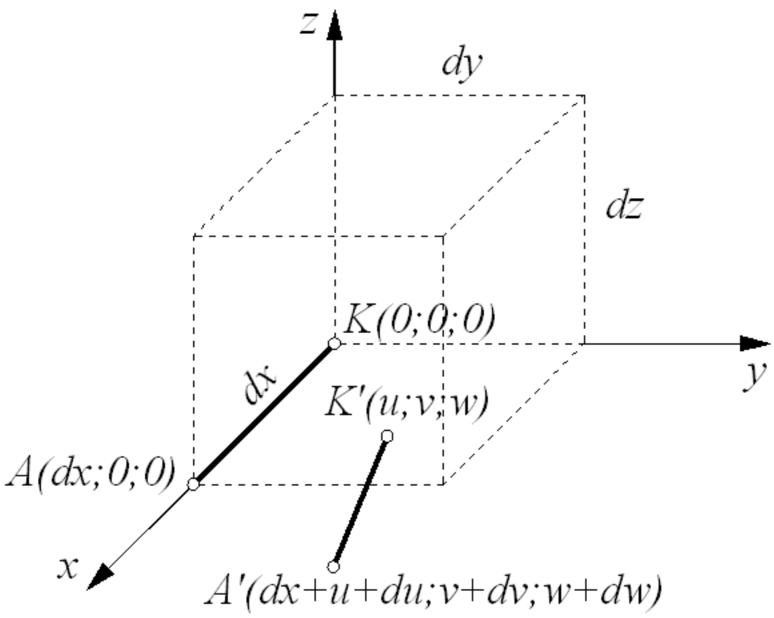


Fig. 6.2: Two positions of AK

The functions of the displacements $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ are continual. Their deformations are presented in this form:

$$d\,u=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz,$$

$$d\,v=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz,$$

$$d\,w=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz.$$

The deformation in the direction of x is considered and that is why the first addend is non-zero. Then the coordinates of a point A’ are $A'\left(dx+u+\frac{\partial u}{\partial x}dx;v+\frac{\partial v}{\partial x}dx;w+\frac{\partial w}{\partial x}dx\right)$. The vector $\overrightarrow{A'K'}$ is $\overrightarrow{A'K'}\left(dx+\frac{\partial u}{\partial x}dx;\frac{\partial v}{\partial x}dx;\frac{\partial w}{\partial x}dx\right)$ and its length is $\overline{A'K'}=dx\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2}$. The second addends are neglected and $\overline{A'K'}\approx dx\left(1+\frac{\partial u}{\partial x}\right)$.

Then the linear deformation in the direction of x is

$$\varepsilon_x = \frac{dx\left(1+\frac{\partial u}{\partial x}\right)-dx}{dx}\ ;\ \varepsilon_x = \frac{\partial u}{\partial x}\,.$$

The formulas of the other linear deformations are deduced by analogous way.

The Cauchy's equations are

$$\varepsilon_x = \frac{\partial u}{\partial x};\quad \gamma_{xy} = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x};$$

$$\varepsilon_y = \frac{\partial v}{\partial y};\quad \gamma_{yz} = \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y};$$

$$\varepsilon_z = \frac{\partial w}{\partial z};\quad \gamma_{zx} = \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\,.$$

6.3. RELATIVE PROLONGATION IN ARBITRARY DIRECTION THROUGH A POINT OF A DEFORMED BODY

Let the direction r is defined by the cosines λ, μ, ν of the unit vector. The relative prolongation ε_r is $\varepsilon_r = \varepsilon_x\lambda^2 + \varepsilon_y\mu^2 + \varepsilon_z\nu^2 + \gamma_{xy}\,\lambda\,\mu + \gamma_{yz}\mu\,\nu + \gamma_{zx}\nu\,\lambda$.

It is evident that it is expressed with the relative prolongations on three perpendicular directions and with the angular deformations.

6.4. PHYSICAL MEANING OF THE ANGULAR DEFORMATIONS γ_{xy}, γ_{yz} AND γ_{zx}.

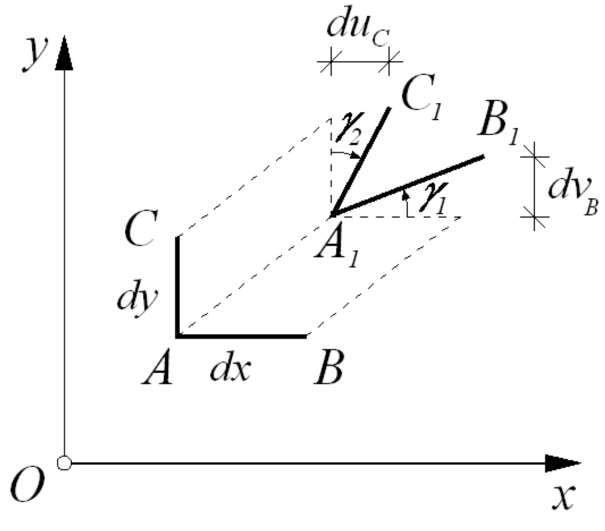


Fig. 6.3: Two positions of the points A, B and C

On demonstration can draw the following conclusions:

$$d\,v_B = \frac{\partial\,v}{\partial\,x}dx;\;\; d\,u_B = \frac{\partial\,u}{\partial\,y}dy;$$

$$\gamma_1 \approx \frac{d\,v_B}{d\,x} = \frac{\partial\,v}{\partial\,x}; \qquad \gamma_2 \approx \frac{d\,u_C}{d\,y} = \frac{\partial\,u}{\partial\,y};$$

$$\gamma_{xy} = \gamma_1 + \gamma_2 = \frac{\partial\,u}{\partial\,y} + \frac{\partial\,v}{\partial\,x}.$$

Analogically are obtained the formulas:

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}; \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}.$$

γ_{xy}, γ_{yz}, γ_{zx} are a measure of the change of the angle between two perpendicular linear elements. Their directions before the deformation are defined with lower indices of γ.

6.5. TENSOR OF THE DEFORMATION

$$T = \begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{vmatrix}$$

The main axes of the deformations coincide with those of the main axes of stresses. For these deformations tensor has the form:

$$T = \begin{vmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{vmatrix}.$$

Invariants of deformation tensor are:

$$I_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_x + \varepsilon_y + \varepsilon_z;$$

$$I_2 = \varepsilon_1\,\varepsilon_2 + \varepsilon_2\,\varepsilon_3 + \varepsilon_3\,\varepsilon_1;$$

$$I_3 = \varepsilon_1\,\varepsilon_2\,\varepsilon_3.$$

6.6. VOLUME DEFORMATION

Volumetric strain is defined as the relative change of the volume of material at a point of a deformable body.

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z = I_1.$$

6.7. SAINT-VENANT EQUATIONS OF THE CONTINUITY OF THE DEFORMATIONS

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}; \qquad \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} \right) = 2\frac{\partial^2 \varepsilon_x}{\partial y\, \partial z};$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial z\, \partial x}; \qquad \frac{\partial}{\partial x}\left(\frac{\partial \gamma z}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) = 2\frac{\partial^2 \varepsilon_y}{\partial z\, \partial x};$$

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{zx}}{\partial y\, \partial z}; \qquad \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} \right) = 2\frac{\partial^2 \varepsilon_z}{\partial x\, \partial y}.$$

CHAPTER 7

DEFLECTION OF BEAMS

7.1. INTRODUCTION

The object of investigation is a straight beam loaded by a set of forces situated in the principal beam plane xz and the important assumption is that the beam axis will belong to the same plane after deformation.

A cantilever beam acted upon by a concentrated force at free end is shown in fig.7.1. The positions of the beam axis before and after deformation are drawn. The Cartesian coordinate system with origin at the fixed support is introduced. The x-axis coincides with the beam axis while the z-axis is directed down. Axes y and z are the *principal axis of the beam*.

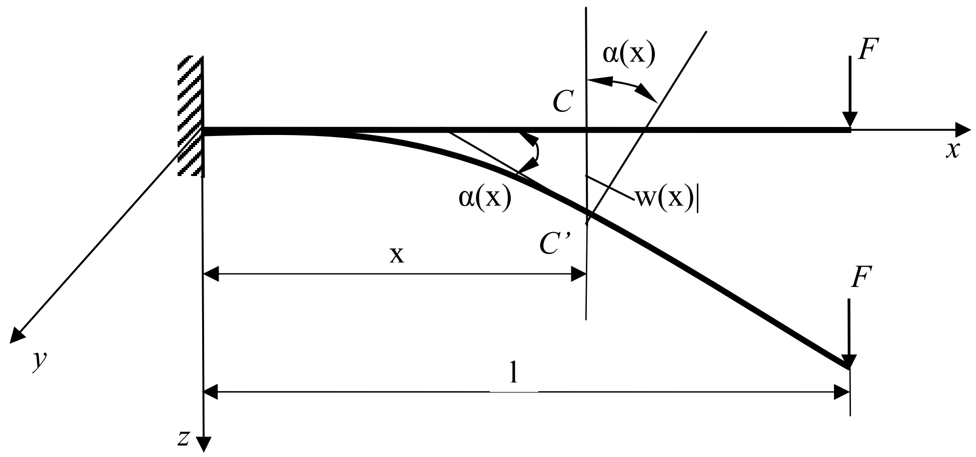


Fig. 7.1: A cantilever beam before and after deformation

A random beam section at a distance x from the fixed support is considered. The section's center of gravity /point C/ occupies the position C' after deformation. Due to the fact that the line CC' is too short relative to the beam length l it is accepted CC' to be perpendicular to the horizontal, i.e. the displacement of the section along the x-axis to be neglected.

By definition *the deflection is the vertical displacement CC' of the beam section's center of gravity. It is perpendicular to the beam axis* and it is denoted by w.

If the deflections are determined for every beam section, then, the new position of the beam axis will be known. The beam axis after deformation is named ***elastic line*** of the beam.

Furthermore, it is accepted the validity of the *Bernoulli's hypothesis* in accordance with which every planar beam section normal to the axis before deformation remains planar and normal to the beam axis after deformation. The two positions of the random beam section at a distance x are shown in fig. 7.1 where the angle $\alpha(x)$ between them is also given.

By definition *the angle of rotation α /the slope/ is the angle between the positions of the beam section considered before and after deformation.*

The problem about the stiffness of the beams subjected to bending is very important in the engineering practice. It is necessary the deformation of the beam to be restricted. In the opposite case the large deflections will influence adversely to the construction as well as the adjacent elements. In the real buildings the beams deflections are considerably smaller than its span. The biggest vertical displacement of the beam section is a function of the length L, for example $L/1000$.

The position of the beam axis after deformation is known when the deflection w and the slope α of the random beam section are determined. These two parameters depend on the coordinate x of this section. It is seen the angle of rotation α is equal to the angle between the tangent in point C' and the x-axis. The angle coefficient in point C' of the beam axis after deformation is $tg\,\alpha = w'$. Because of the small angle of rotation α it can be supposed $tg\,\alpha \approx \alpha$. Thus, *the relation between the functions of the deflection $w(x)$ and slope $\alpha(x)$ is:*

$$\alpha = w'. \tag{7.1}$$

Then, the conclusion that the beam axis position after deformation is completely known when the function $w(x)$ has been derived is made.

7.2. THE DERIVATION OF THE ELASTIC LINE DIFFERENTIAL EQUATION FOR A STRAIGHT BEAM

A beam loaded by a set of forces situated in a vertical plane xz is investigated (fig.7.1). Thus, the bending moment obtained is along y-axis. When *the straight beam of constant cross-section is subjected to special bending*, it bends about principal axis y under the action of moment M_y and, then, the curvature of the beam axis is:

$$\kappa = \frac{1}{R} = \frac{M_y}{EI_y}. \tag{7.2}$$

Here, E is the modulus of elasticity, R is the curvature radius after deformation, I_y is the principal moment of inertia about y-axis. The product EI_y is named the stiffness of the beam subjected to bending.

If the transverse forces act upon a beam, then the beam axis will not bend in an arc.

It is allowed the equation (7.2) takes part for every beam section at which the bending moment M_y acts. It is obvious for the beam of the constant stiffness EI_y, when the bending moment M_y changes, then, the radius of curvature R changes, too.

The axis z in fig. 7.1 is directed down. *It is convenient because the loadings in the real problems cause the vertical displacements having downward sense.*

The elastic line curvature κ in point C' can be expressed by the function $w(x)$ applying the well-known equation of Mathematics:

$$\kappa = \frac{1}{R} = \frac{\dfrac{d^2w}{dx^2}}{\pm\left[1+\left(\dfrac{dw}{dx}\right)^2\right]^{3/2}}. \tag{7.3}$$

The strict equation of the elastic line is obtained by the comparison of the right-hand sides of (7.2) and (7.3).

$$\frac{\dfrac{d^2w}{dx^2}}{\pm\left[1+\left(\dfrac{dw}{dx}\right)^2\right]^{3/2}} = \frac{M_y}{EI_y}. \tag{7.4}$$

This is non-linear differential equation which strict solution is very complex. Because of that the equation is rearranged using the condition of the small deformations in a beam – the angle of rotation $\alpha = \frac{dw}{dx} = w'$ has the values from 0,001 to 0,01 rad. Furthermore, the values of $\left(\frac{dw}{dx}\right)^2$ in the denominator will be much smaller compared to one. Then, the equation (7.4) can be written in form:

$$\pm \frac{d^2 w}{d x^2} = \frac{M_y}{EI_y}. \tag{7.5}$$

Next step is to obtain the correct sign in the equation's left-hand side. The elements of the beam after deformation are given in fig. 7.2 where the left sketch shows the element subjected to *positive bending moment* while the right one shows the element subjected to *negative moment*.

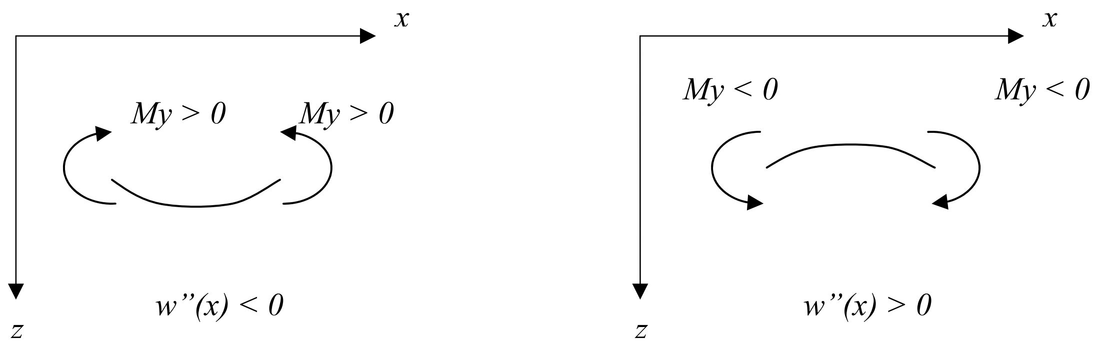


Fig. 7.2: The beam elements subjected to bending moment M_y after deformation M_y

It is known from the Mathematics, if the function's second derivative is *positive*, then the function's graph is *concave* and vice versa. Thus, analyzing fig. 7.2, the conclusion that the functions $M_y(x)$ and $w''(x) = \frac{d^2 w}{d x^2}$ always have opposite signs can be made.

Finally*, the approximate differential equation of elastic line is obtained*:

$$EI_y w''(x) = -M_y(x). \tag{7.6}$$

There are some different methods for the determination of the vertical displacements in beams.

7.3. DIRECT INTEGRATION METHOD

This method is applicable in the cases when the whole elastic line of the beam must be found.

7.3.1. BEAMS OF *CONSTANT* CROSS-SECTION

The stiffness EI_y has constant value along the beam length. Then, only w'' and M_y depend on x in (7.6). This equation can be easily integrated and the function $w(x)$ as well as function $\alpha(x)$ can be directly obtained. The integration is:

$$EI_y w'(x) = EI_y \alpha(x) = -\int M_y(x)\, dx + C_1; \tag{7.7}$$

$$EI_y w(x) = -\int \left[\int M_y(x)\, dx\right] dx + C_1 x + C_2. \tag{7.8}$$

The constants C_1 and C_2 will be derived by the *kinematical boundary conditions*. They correspond to the constraints of the beam and they are related to the deflection w and the slope α.

a) Simply supported beam acted upon by two moments (Fig.7.3)

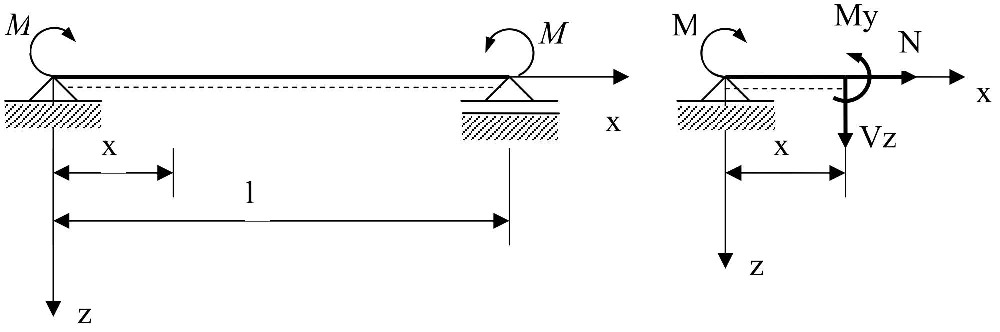


Fig. 7.3: **Simply supported beam upon which two moments act**

The bending moment is constant $M_y = M$. Then, the differential equation (7.6) takes a form:

$$EI_y w''(x) = -M \; . \tag{7.9}$$

The functions $\alpha(x)$ and $w(x)$ are derived by integration:

$$EI_y\, w'(x) = EI_y\, \alpha(x) = -M\, x + C_1 \, ; \tag{7.10}$$

$$EI_y w(x) = -0{,}5 M\, x^2 + C_1\, x + C_2 \, . \tag{7.11}$$

To obtain the integration constants C_1 and C_2 *the kinematical boundary conditions* are used – the vertical displacements in the both beam ends are equal to zero:

$$w\,(0) = 0;\; w\,(l) = 0 \; . \tag{7.12}$$

Then:

$$C_2 = 0;\; C_1 = 0{,}5Ml \; . \tag{7.13}$$

Finally, the functions $\alpha(x)$ and $w(x)$ are:

$$EI_y\, \alpha(x) = -M.x + 0{,}5Ml \, ; \tag{7.14}$$

$$EI_y w(x) = -0{,}5\, M\, x^2 + 0{,}5Ml\, x \, . \tag{7.15}$$

b) Simply supported beam acted upon by a distributed load of intensity q **(Fig.7.4)**

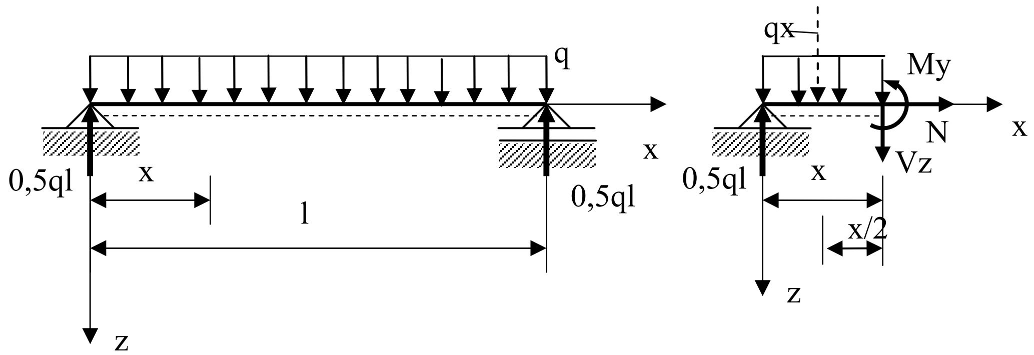


Fig. 7.4: **Simply supported beam acted upon by a distributed load of intensity** q

The bending moment function is:

$$M_y(x) = -\frac{1}{2}x^2 + \frac{1}{2}ql\,x \tag{7.16}$$

The elastic line differential equation (7.6) is:

$$EI_y w''(x) = \frac{1}{2}q\,x^2 - \frac{1}{2}ql\,x \, . \tag{7.17}$$

The functions of slope $\alpha(x)$ and deflection $w(x)$ are carried out by integration:

$$EI_y\, w'(x) = EI_y\, \alpha(x) = \frac{1}{6} q\, x^3 - \frac{1}{4} ql\, x^2 + C_1 ; \tag{7.18}$$

$$EI_y\, w(x) = \frac{1}{24} qx^4 - \frac{1}{12} ql\, x^3 + C_1 x + C_2 . \tag{7.19}$$

The integration constants are:

$$C_2 = 0;\, C_1 = \frac{1}{24} q\, l^3 . \tag{7.20}$$

Finally:

$$EI_y\, \alpha(x) = \frac{1}{6} q\, x^3 - \frac{1}{4} ql\, x^2 + \frac{1}{24} q\, l^3 ; \tag{7.21}$$

$$EI_y\, w(x) = \frac{1}{24} qx^4 - \frac{1}{12} ql\, x^3 + \frac{1}{24} q\, l^3 x . \tag{7.22}$$

c) Cantilever beam loaded by a vertical force F at free end (Fig.7.5)

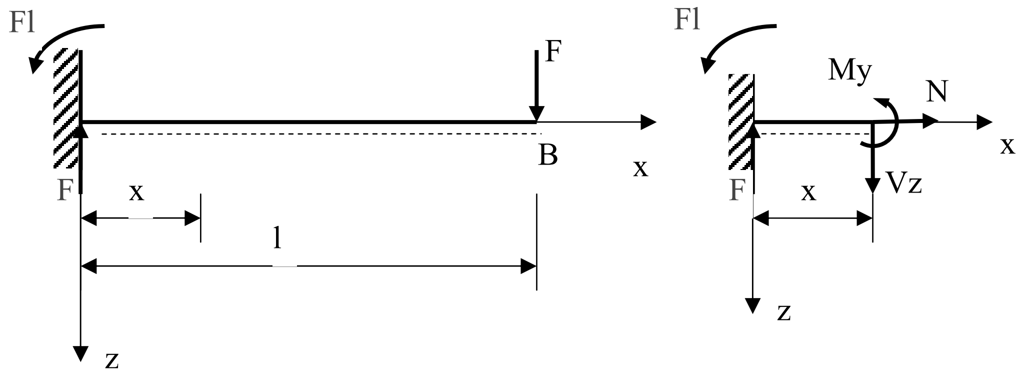


Fig. 7.5: Cantilever beam loaded by a vertical force F at free end

Here, the function of M_y is

$$M_y(x) = -F * (l - x) \tag{7.23}$$

and, consequently, $EI_y w''(x) = F * (l - x)$. (7.24)

The functions of $\alpha(x)$ and $w(x)$ are:

$$EI_y\, w'(x) = EI_y\, \alpha(x) = -\frac{1}{2} F\, x^2 + Fl\, x + C_1 ; \tag{7.25}$$

$$EI_y\, w(x) = -\frac{1}{6} Fx^3 + \frac{1}{2} Fl\, x^2 + C_1 x + C_2 . \tag{7.26}$$

The *kinematical boundary conditions* relates to the rotation and displacement of the fixed support which are equal to zero, i.e.

$$w(0) = 0;\, w'(0) = 0 . \tag{7.27}$$

Thus, the constants are

$$C_2 = 0;\, C_1 = 0 . \tag{7.28}$$

Then, the expressions (7.25) and (7.26) take the form:

$$EI_y\, \alpha(x) = -\frac{1}{2} F\, x^2 + Fl\, x ; \tag{7.29}$$

$$EI_y\, w(x) = -\frac{1}{6} Fx^3 + \frac{1}{2} Fl\, x^2 . \tag{7.30}$$

The vertical displacement of the free beam end is $w_B = \dfrac{Fl^3}{3EI_y}$.

d) Cantilever beam loaded by a vertical uniformly distributed load of intensity q (Fig.7.6)

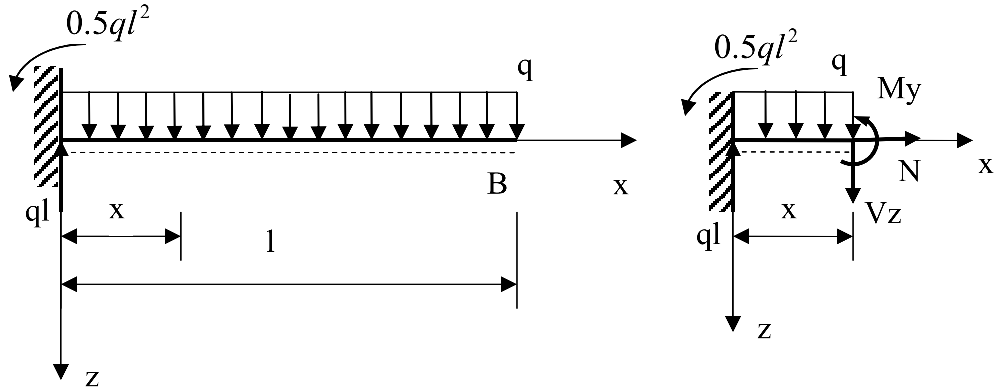


Fig. 7.6: Cantilever beam loaded by a vertical uniformly distributed load of intensity q

The bending moment function is

$$M_y(x) = -\frac{1}{2} q (l - x)^2 , \tag{7.31}$$

and the differential equation of the elastic line takes the form:

$$EI_y w''(x) = \frac{1}{2} q (l - x)^2 . \tag{7.32}$$

Further, the functions of the slope and deflection are obtained:

$$EI_y\, w'(x) = EI_y\, \alpha(x) = \frac{1}{6} q\, x^3 - \frac{1}{2} ql\, x^2 + \frac{1}{2} ql^2\, x + C_1 ; \tag{7.33}$$

$$EI_y\, w(x) = \frac{1}{24} qx^4 - \frac{1}{6} ql\, x^3 + \frac{1}{4} ql^2\, x^2 + C_1 x + C_2 . \tag{7.34}$$

The *kinematical boundary conditions* are the same like the case earlier. Because of that $C_1 = 0 ; C_2 = 0$.

Finally:

$$EI_y\, \alpha(x) = \frac{1}{6} q\, x^3 - \frac{1}{2} ql\, x^2 + \frac{1}{2} ql^2\, x ; \tag{7.35}$$

$$EI_y\, w(x) = \frac{1}{24} qx^4 - \frac{1}{6} ql\, x^3 + \frac{1}{4} ql^2\, x^2 . \tag{7.36}$$

The vertical displacement of the free beam end is $w_B = \dfrac{ql^4}{8EI_y}$.

e) Beam containing two or more segments with different equations of the bending moment

Because of the different bending moment functions the functions of $w(x)$ and $\alpha(x)$ are also different for each segment. If n is the number of the segments, then the number of the integration constants will be *2n*. To find them, the *kinematical boundary conditions* must be written in accordance with the supports as well as the segment's boundary points. Some boundary conditions are given in Table 7.1.

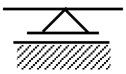
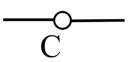
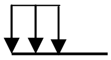
	$w_C^{ляво} = w_C^{дясно} = 0; \quad \alpha_C^{ляво} = \alpha_C^{дясно}$
	$w_C^{ляво} = w_C^{дясно}$
	$w_C^{ляво} = w_C^{дясно}; \alpha_C^{ляво} = \alpha_C^{дясно}$

Table 7.1

Then, the expressions of $w(x)$ and $\alpha(x)$ for all of the segments are written and the elastic line of entire beam is obtained.

Problem 7.3.1: The beam shown in fig. 7.7 subjected to a vertical force F has constant stiffness EI_y. The lengths of the two segments are given as functions of the parameter a. Determine the functions of the slope $\alpha(x)$ and deflection $w(x)$ in the two segments of the beam applying *the direct integration method*.

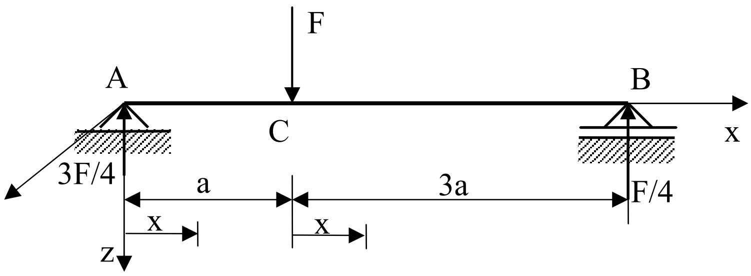
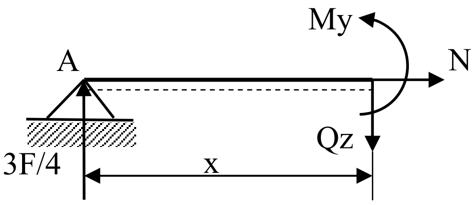


Fig. 7.7: Simply supported beam acted upon by a single vertical force

The equation of the bending moment M_y must be determined for each segment. After that, it has to be substituted in equation (7.6) and must be integrated two times.

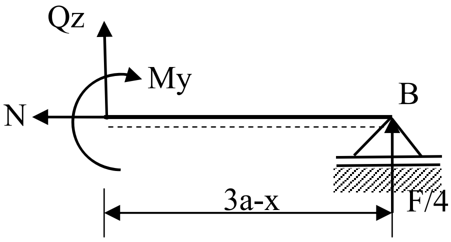
Segment AC: $0 \le x \le a$



$M_y(x) = \dfrac{3F}{4}x$; $\quad EI_y w_1''(x) = -\dfrac{3F}{4}x$;

$EI_y w_1'(x) = EI_y \alpha_1(x) = -\dfrac{3F}{8}x^2 + C_1$; $EI_y w_1(x) = -\dfrac{F}{8}x^3 + C_1 x + C_2$.

Segment CB: $0 \le x \le 3a$



$$M_y(x) = -\frac{F}{4}x + \frac{3Fa}{4};$$

$$EI_y w_2''(x) = \frac{F}{4}x - \frac{3Fa}{4};$$

$$EI_y w_2'(x) = EI_y \alpha_2(x) = \frac{F}{8}x^2 - \frac{3Fa}{4}x + C_3;$$

$$EI_y w_2(x) = \frac{F}{24}x^3 - \frac{3Fa}{8}x^2 + C_3 x + C_4.$$

The *kinematical boundary conditions* necessary to obtain the integration constants are four. Two of them are related to the supports – the vertical displacements are not possible, i.e

1) $w_1(0) = 0$; 2) $w_2(3a) = 0$.

The others are written with respect to the beam section C, which is a boundary section. At this section the vertical displacements and the angles of rotation in the left and in the right are equal:

3) $w_1(a) = w_2(0)$; 2) $\alpha_1(a) = \alpha_2(0)$.

The expressions for the integration constants determination are:

1) $C_2 = 0$;

2) $\frac{F}{24}(3a)^3 - \frac{3Fa}{8}(3a)^2 + C_3 3a + C_4 = 0$;

3) $-\frac{F}{8}a^3 + C_1 a = C_4$;

4) $-\frac{3F}{8}a^2 + C_1 = C_3$,

and the constants are obtained $C_1 = \frac{7Fa^2}{8}$; $C_2 = 0$; $C_3 = \frac{Fa^2}{2}$; $C_4 = \frac{3Fa^3}{4}$.

Finally, the functions of $w(x)$ and $\alpha(x)$ in the two segments take form:

$$EI_y \alpha_1(x) = -\frac{3F}{8}x^2 + \frac{7Fa^2}{8}; \quad EI_y w_1(x) = -\frac{F}{8}x^3 + \frac{7Fa^2}{8}x;$$

$$EI_y \alpha_2(x) = \frac{F}{8}x^2 - \frac{3Fa}{4}x + \frac{Fa^2}{2}; \quad EI_y w_2(x) = \frac{F}{24}x^3 - \frac{3Fa}{8}x^2 + \frac{Fa^2}{2}x + \frac{3Fa^3}{4}.$$

7.3.2. BEAMS OF *TAPERED* CROSS-SECTION

The cross-section can change *smooth* or *in steps* along the beam length. Then, the moment of inertia I_y is not a constant, but a function of x.

The differential equation (7.6) has the form:

$$EI_y(x) w''(x) = -M_y(x). \qquad (7.37)$$

a) Beams with *smooth change* of the cross-section

The expressions of $\alpha(x)$ and $w(x)$ are obtained by integration of (7.37):

$$\alpha(x)=\frac{dw}{dx}=-\frac{1}{E}\int\frac{M_y(x)}{I_y(x)}dx+C_1; \tag{7.38}$$

$$w(x)=-\frac{1}{E}\int\left[\int\frac{M_y(x)}{I_y(x)}dx\right]dx+C_1 x+C_2. \tag{7.39}$$

The solution's procedure is similar like this one of the beam of constant cross-section. If $\frac{M_y(x)}{I_y(x)}$ and $\int\frac{M_y(x)}{I_y(x)}dx$ are functions which can be integrated, then, $\alpha(x)$ and $w(x)$ can be determined. In opposite case, the method for their approximate calculation must be used.

Problem 7.3.2.1: The cantilever beam of length l shown in fig. 7.8 is acted upon by a single vertical force F applied at free beam end. The law of change of the beam's moment of inertia is $I_y(x)=I\frac{x+l}{l}$ while the moment of inertia in section A is a constant $I_y=I$. The modulus of elasticity E is also given.

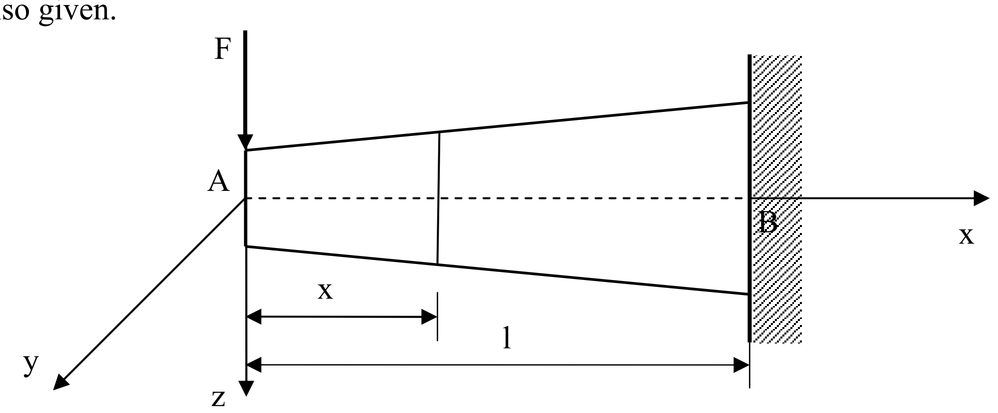


Fig.7.8: Beam with smooth change of the cross-section along its length

In this case, the differential equation of the elastic line has a form (7.37). It is obvious the bending moment is $M_y(x)=-F\,x$ and (7.37) becomes:

$$w''(x)=\frac{Fx}{EI\frac{x+l}{l}} \quad \text{or} \quad \frac{d^2w(x)}{dx^2}=\frac{Fl}{EI}\frac{x}{x+l}.$$

First, the two sides of this equation are multiplied by x. Then, the integration from A to B is made: $\int_A^B x\frac{d^2w(x)}{dx^2}dx=\frac{Fl}{EI}\int_A^B\frac{x^2}{x+l}dx$. The integrals in the left-hand side must be solved separately:

$$\left(x\frac{dw}{dx}\right)_A^B-\int_A^B\frac{dw(x)}{dx}dx=\frac{Fl}{EI}\int_A^B\frac{x^2}{x+l}dx.$$

The solution continues with rearrangements as follows:

$x_B\alpha_B-x_A\alpha_A-(w_B-w_A)=\frac{Fl}{EI}\int_A^B\frac{(x^2-l^2)+l^2}{x+l}dx$. Further, taking into account $x_A=0$, $w_B=0$ and $\alpha_B=0$, the expression $w_A=\frac{Fl}{EI}\left[\int_0^l(x-l)\,dx+l^2\int_0^l\frac{dx}{x+l}\right]$ is obtained. Finally, $w_A=\frac{Fl^3}{EI}[\ln(l)-0{,}5]$.

b) Beams of cross-section which changes *in steps*

The beam of n segments with different moments of inertia I_{yi}, $i = 1,2,...,n$ is considered. In this case, one of the moments of inertia, for example $I_0 = I_{y1}$, must be chosen as a *basic* one. Further, EI_0 -multiple value of the displacement for every segment must be determined.

The differential equation which has to be integrated for the segment with moment of inertia I_{y1} is:

$$EI_0\, w_1''(x) = -M_{y1}(x), \qquad (7.40)$$

while for the i-th segment it is:

$$EI_0\, w''(x) = -\frac{I_0}{I_{yi}} M_{yi}(x), \qquad i = 2,3,...,n\,. \qquad (7.41)$$

Problem 7.3.2.2: The beam shown in fig. 7.9 has two segments which lengths are functions of the parameter a. The segment AB's moment of inertia is $I_y = I_1$ and the ratio $I_1 / I_2 = 2$. The modulus of elasticity is E. Apply the direct integration method to obtain the functions $\alpha(x)$ and $w(x)$ for the two beam segments.

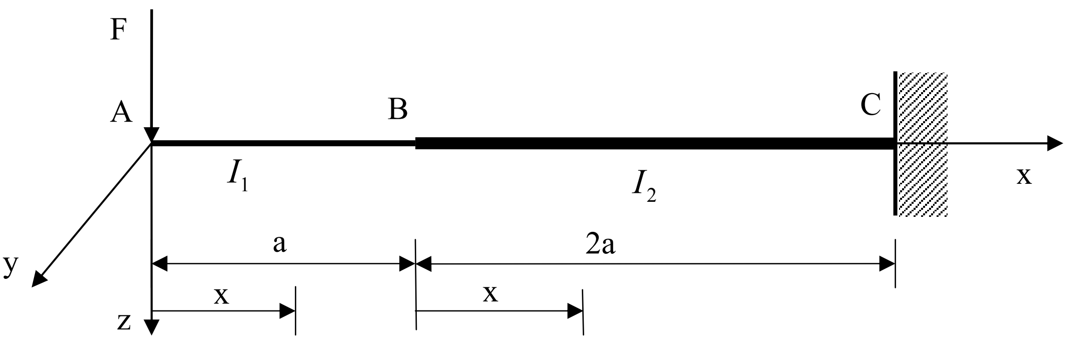
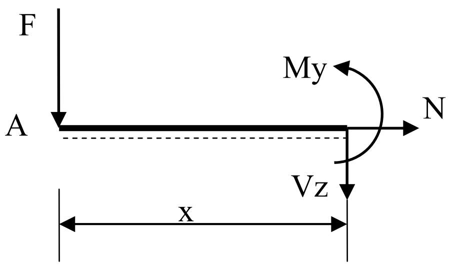


Fig. 7.9: Beam of cross-section which changes in steps along its length

The *essential* moment of inertia $I_0 = I_1$ is accepted.



Segment AB: $0 \le x \le a$

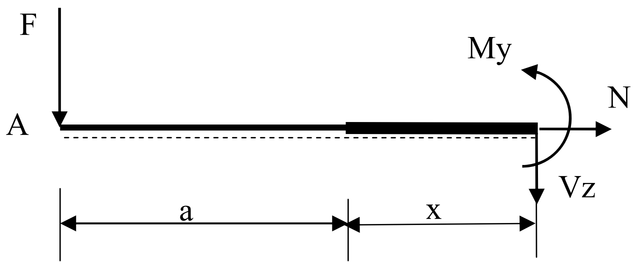
$M_y(x) = -Fx$;

$EI_0 w_1''(x) = Fx$;

$EI_0 w_1'(x) = EI_0 \alpha_1(x) = \frac{F}{2}x^2 + C_1$;

$EI_0 w_1(x) = \frac{F}{6}x^3 + C_1 x + C_2$.

Segment BC: $0 \le x \le 2a$



$M_y(x) = -F(x+a)$; $\quad EI_0 w_2''(x) = \frac{I_0}{I_2} F(x+a) = 2F(x+a) = 2Fx + 2Fa$;

$EI_0 w_2'(x) = EI_o \alpha_2(x) = Fx^2 + 2Fax + C_3$; $EI_0 w_2(x) = \frac{F}{3}x^3 + Fax^2 + C_3 x + C_4$.

The *kinematical boundary conditions* are:

1) $w_2(2a) = 0$; $\frac{F}{3}(2a)^3 + Fa(2a)^2 + C_3 2a + C_4 = 0$;

2) $\alpha_2(2a) = 0$; $F(2a)^2 + 2Fa2a + C_3 = 0$;

3) $w_1(a) = w_2(0)$; $\frac{F}{6}a^3 + C_1 a + C_2 = C_4$;

4) $\alpha_1(a) = \alpha_2(0)$; $\frac{F}{2}a^2 + C_1 = C_3$.

It is obtained $C_1 = -\frac{17Fa^2}{2}$; $C_2 = \frac{53Fa^3}{3}$; $C_3 = -8Fa^2$; $C_4 = \frac{28Fa^3}{3}$.

Then, the EI_0-multiple values of the functions $w(x)$ and $\alpha(x)$ are:

$EI_0 \alpha_1(x) = \frac{F}{2}x^2 - \frac{17Fa^2}{2}$; $EI_0 w_1(x) = \frac{F}{6}x^3 - \frac{17Fa^2}{2}x + \frac{53Fa^3}{3}$;

$EI_o \alpha_2(x) = Fx^2 + 2Fax - 8Fa^2$; $EI_0 w_2(x) = \frac{F}{3}x^3 + Fax^2 - 8Fa^2 x + \frac{28Fa^3}{3}$.

7.4. MOHR'S ANALOGY METHOD

The direct integration method is convenient when the equation of the beam elastic line must be obtained. However, if the beam contains many segments, then the application of this method is very clumsy. Besides, in many practical cases only the deflection and slope of definite beam section have to be determined. For such cases, the *Mohr*'s analogy method is developed[1].

The essence of the method is: The differential equation of the elastic line is written $EI_y w''(x) = -M_y(x)$. Under the real beam the second beam of the same length, named *fictitious* is drawn. The bending moment diagram of the real beam becomes the distributed load of intensity $\overline{q}$ upon the fictitious beam. If *the real beam bending moment diagram is positive, then the load* $\overline{q}$ *is directed along the positive sense of* $z-$ *axis and vice versa*. The supports of the fictitious beam are indefinite. It can be noted that they have the support reactions equalizing the load $\overline{q}$.

The magnitude of the bending moment in every section of the fictitious beam can be carried out $\overline{M}$. The familiar differential relation $\overline{M}''(x) = -\overline{q}(x)$ takes part. It is juxtaposed to the elastic line differential equation and it is accepted $\overline{q}(x) = M_y(x)$. Thus, the relation $EI_y w''(x) = \overline{M}''(x)$ is obtained. After its integration it is determined $EI_y w(x) = \overline{M}(x)$. Next step is the differentiation of the equation above. After that, taking into account $\alpha(x) = w'(x)$ and $\overline{V}(x) = \overline{M}'(x)$, it is obtained $EI_y \alpha(x) = \overline{V}(x)$.

Finally, the formulas for determination of the deflection w and the slope α in definite section of the real beam are:

$$w = \frac{\overline{M}}{EI_y}; \; \alpha = \frac{\overline{V}}{EI_y}. \tag{7.53}$$

The deflection w *of the real beam section is equal to the ratio between the bending moment in the same section of the fictitious beam and the real beam stiffness* EI_y. *The slope* α *of the real beam*

[1] *Christian Otto Mohr* (1835-1918) is a German engineer, professor on Mechanics in Stuttgart and Dresden

section is equal to the ratio between the shearing force in the same section of the fictitious beam and the real beam stiffness EI_y.

These relations lead to the conclusions, as follows:

- If the deflection w is equal to zero in some beam section, then the bending moment $\overline{M}$ in the same section of the fictitious beam must be equal to zero, too.
- If the slope α is equal to zero in some beam section, then the shearing force $\overline{V}$ in the same section of the fictitious beam must be equal to zero, too.
- If the deflection w and the slope α are different than zero in some beam section, then the bending moment $\overline{M}$ and the shearing force $\overline{V}$ in the same section of the fictitious beam must be different than zero, too.

The correspondences between the supports of the real and fictitious beams are shown in the next table:

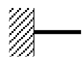
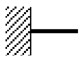
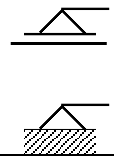
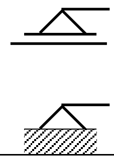
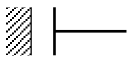
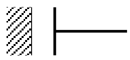
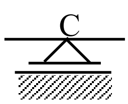
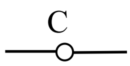
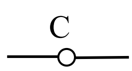
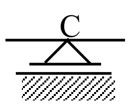
Real beam		***Fictitious*** beam	
	$w \neq 0; \alpha \neq 0$		$\overline{M} \neq 0; \overline{V} \neq 0$
	$w = 0; \alpha = 0$		$\overline{M} = 0; \overline{V} = 0$
	$w = 0; \alpha \neq 0$		$\overline{M} = 0; \overline{V} \neq 0$
	$\alpha = 0$		$\overline{V} = 0$
	$w_c^{left} = w_C^{right} = 0$; $\alpha_C^{left} = \alpha_C^{right}$		$\overline{M}_C^{left} = \overline{M}_C^{right} = 0$; $\overline{V}_C^{left} = \overline{V}_C^{right}$
	$w_C^{left} = w_C^{right}$; $\alpha_C^{left} \neq \alpha_C^{right}$		$\overline{M}_C^{left} = \overline{M}_C^{right}$; $\overline{V}_C^{left} \neq \overline{V}_C^{right}$

Table 7.2

It can be noted that the statically determinate fictitious beam corresponds *always* to the statically determinate real beam.

The *Mohr*'s analogy method is appropriate for the determination of the vertical displacement w and the angle of rotation α of the definite beam section. Their determination follows the steps:

- The bending moment diagram of the real beam has to be built.
- The fictitious beam is drawn according to the correspondences in the table above.

- The bending moment diagram of the real beam is put as a distributed load $\overline{q}$ upon the fictitious beam. $\overline{q}$ has dimension kNm .
- The fictitious beam support reactions must be determined.
- If the deflection w of the real beam section K must be obtained, then the magnitude of the bending moment $\overline{M_K}$ in the same section of the fictitious beam must be determined first. It has dimension kNm^3 . Thus, the deflection is $w_K = \frac{\overline{M_K}}{EI_y}$.
- If the slope α of the real beam section K must be obtained, then the magnitude of the shearing force $\overline{V_K}$ in the same section of the fictitious beam must be determined first. It has dimension kNm^2 . Thus, the slope is $\alpha_K = \frac{\overline{V_K}}{EI_y}$.

Problem 7.4.1: A cantilever beam of length l and stiffness EI_y is subjected to a single vertical force F at free end. Apply the *Mohr*'s analogy method to find the vertical displacement w_A and the angle of rotation α_A of the beam free end.

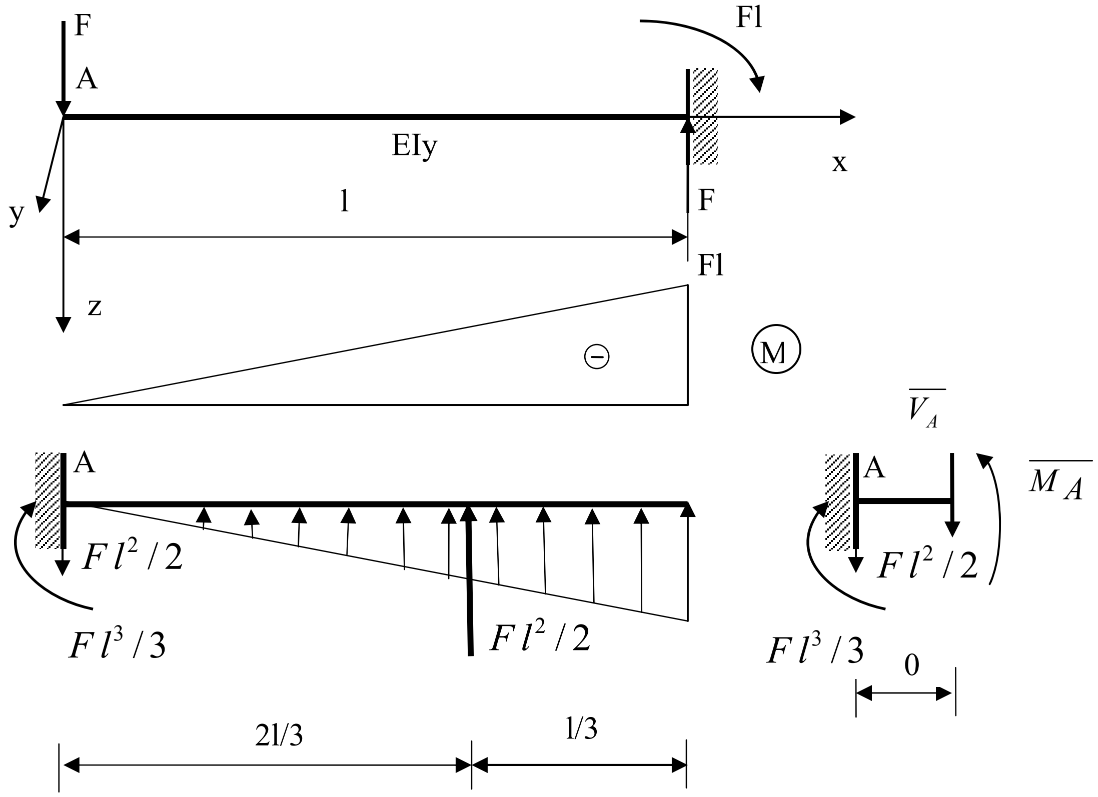


Fig.7.12: A cantilever beam subjected to a single vertical force at free end

First, the real beam bending moment diagram is drawn. After that, the fictitious beam acted upon by a distributed load corresponding to the bending moment diagram is built. Then, to find the internal forces $\overline{V_A}$ and $\overline{M_A}$ in section A of the fictitious beam, the equilibrium of the left part cut is considered and the equation $\sum V = 0$ is written. It is obtained $\overline{V_A} = -\frac{Fl^2}{2}$. According to the *Mohr's* analogy method $EI_y \alpha_A = \overline{V_A}$ takes part. Thus, the slope in section A is $\alpha_A = -\dfrac{Fl^2}{2EI_y}$. Further, applying the equilibrium equation $\sum M = 0$, the bending moment is calculated $\overline{M_A} = \dfrac{Fl^3}{3}$. Finally, taking into account $EI_y w_A = \overline{M_A}$, the deflection is found $w_A = \dfrac{Fl^3}{3EI_y}$.

Problem 7.4.2: A cantilever beam of length l and stiffness EI_y is loaded by a uniformly distributed vertical load of intensity q. Determine the vertical displacement w_C and the slope α_C using the *Mohr*'s analogy method.

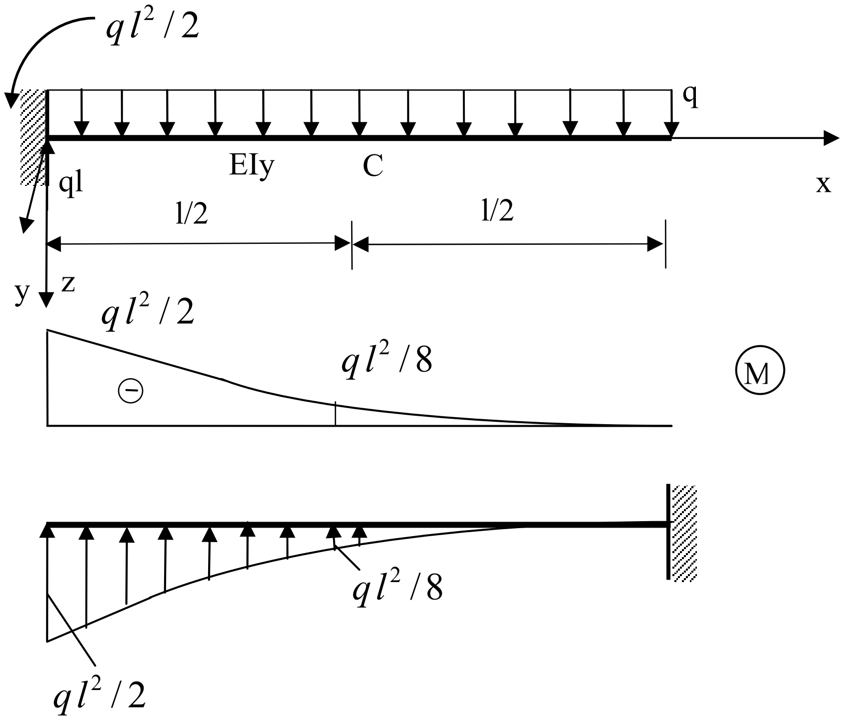


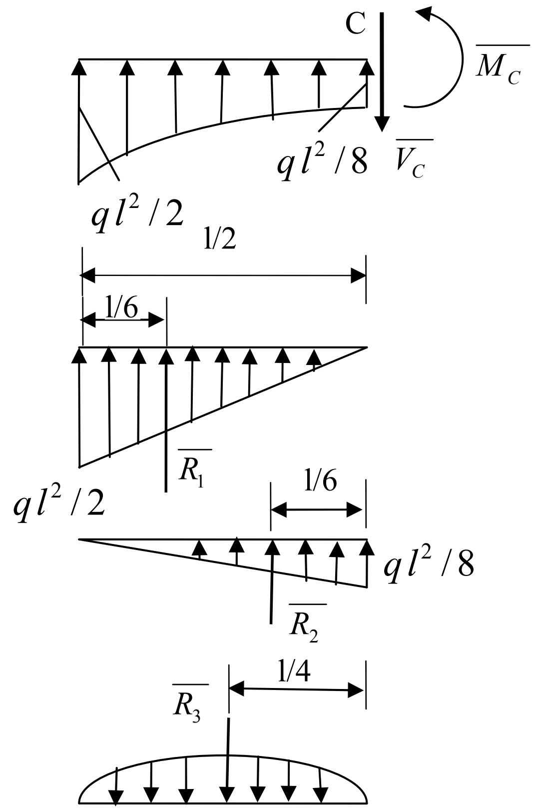
Fig.7.12: A cantilever beam loaded by a uniformly distributed vertical load

The bending moment diagram of the real beam is built. Then, the fictitious beam acted upon by a distributed load corresponding to the real beam bending moment diagram is drawn. To find the internal forces of the section C, the left beam part is chosen for investigation.

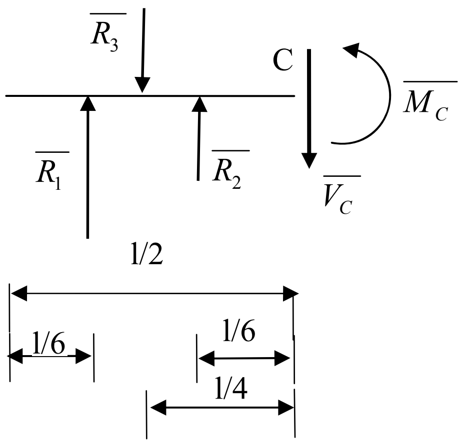
The distributed load has a complex shape and it is convenient to be resolved as shown. The resultant forces of the distributed load are obtained:

$$\overline{R_1} = \frac{ql^3}{8};\quad \overline{R_2} = \frac{ql^3}{32};\quad f = \frac{q\left(\frac{l}{2}\right)^2}{8} = \frac{ql^2}{32};\quad \overline{R_3} = \frac{2}{3} f \frac{l}{2} = \frac{ql^3}{96}.$$

All quantities relating to the fictitious beam are labeled by bar.



Further, the equilibrium equations of the beam part considered are written.



$$\sum V = 0;$$

$$\overline{R}_1 + \overline{R}_2 - \overline{R}_3 - \overline{V}_C = 0 \ ; \quad \frac{ql^3}{8} + \frac{ql^3}{32} - \frac{ql^3}{96} - \overline{V}_C = 0;$$

$$\overline{V}_C = \frac{7ql^3}{48}.$$

$$\sum M = 0;$$

$$-\overline{R}_1 \frac{l}{3} - \overline{R}_2 \frac{l}{6} + \overline{R}_3 \frac{l}{4} + \overline{M}_C = 0;$$

$$\overline{M}_C = \frac{17ql^4}{384}.$$

Then, the analogy $EI_y \alpha_C = \overline{V}_C$ и $EI_y w_C = \overline{M}_C$ is applied. Finally, the vertical displacement and the angle of rotation of section C are obtained: $w_C = \dfrac{17ql^4}{384EI_y}$, $\alpha_C = \dfrac{7ql^3}{48EI_y}$.

In the case of a beam containing the segments of *different moments of inertia*, to apply the *Mohr'*s analogy method, one of the moments of inertia must be chosen for *basic* one, first. Then, for each segment, the distributed load $\overline{q}$ upon the fictitious beam corresponding to the real beam bending moment diagram must be multiplied by the ratio between the basic moment of inertia and the moment of inertia of the segment considered. Generally, it can be written for the i-th segment $\overline{q}_i = M_i \dfrac{I_O}{I_i}$.

When the distributed load is parabola, its maximum /in the middle/ can be calculated by the formula: $f = \frac{ql^2}{8}\frac{I_O}{I_i}$. Here, I_i is the moment of inertia of the segment upon which the distributed load in the shape of parabola acts.

If the beam has *tapered* section along its length, i.e. the moment of inertia is a function $I(x)$, I_O /*basic*/ will be the moment of inertia of the definite section. Then, the fictitious load will be found by the equation: $\bar{q} = M(x)\frac{I_0}{I(x)}$.

The bending moment and the shearing force in the random section of the fictitious beam are related to the deflection and slope in the same section of the real beam by formulas:
$\overline{M} = EI_O w$; $\overline{V} = EI_O \alpha$.

Problem 7.4.3: A steel beam is supported and loaded, as shown. The moment of inertia in the segment AB is $I_1 = 11620\,cm^4$, while the moment of inertia in the segment BC is $I_2 = 5500\,cm^4$. The modulus of elasticity is $E = 20000\,kN/cm^2$. Apply the *Mohr*'s analogy method to determine the slope α of the beam section A and the deflection w of the beam section B.

The bending moment diagram of the Gerber beam considered is built and the fictitious beam is drawn under it. The fictitious beam in the segment AB which has the moment of inertia $I_1 = 11620\,cm^4$ is loaded by a distributed load corresponding to the bending moment diagram in the same segment. In the segment BC the load values are multiplied by the ratio $\frac{I_1}{I_2} = \frac{11620}{5500} = 2{,}113$, because the formula $\bar{q} = M\frac{I_1}{I_2}$ takes part.

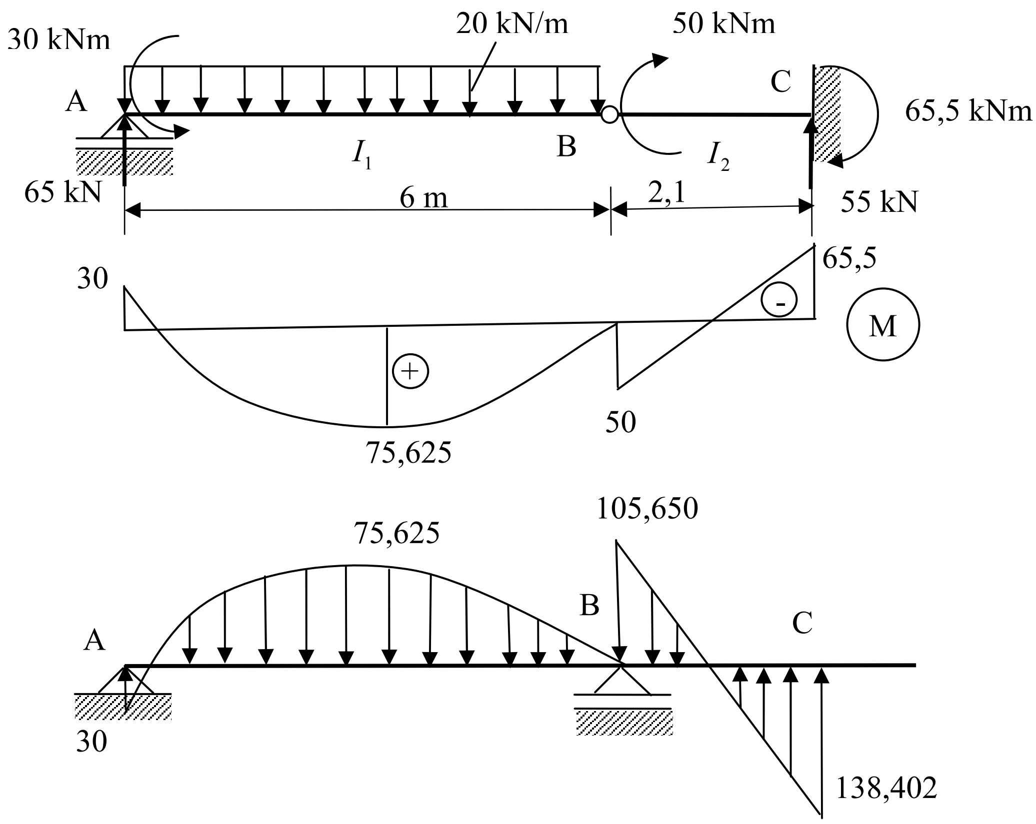
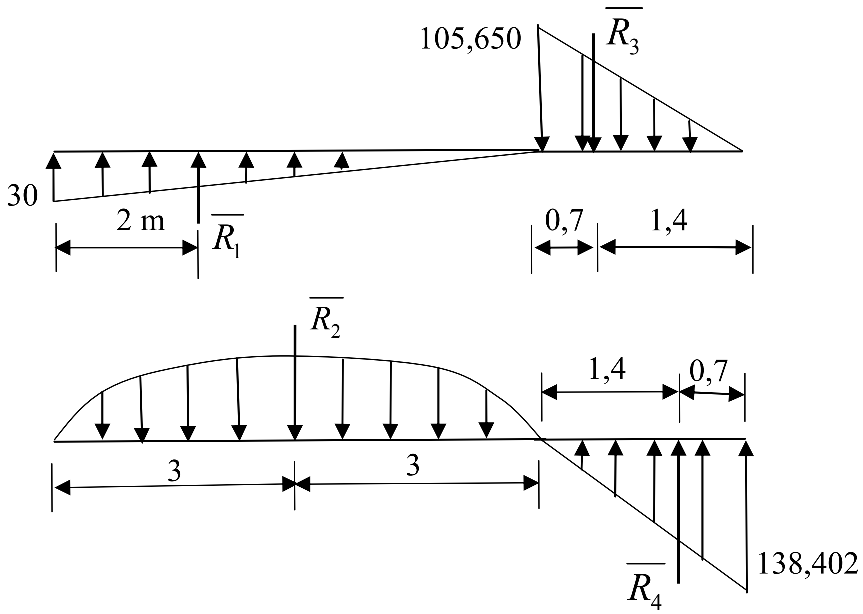


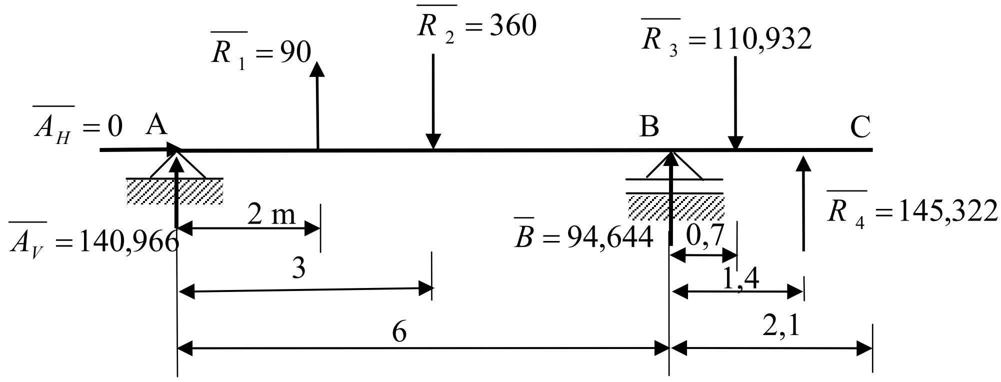
Fig.7.13: Gerber beam

After that, the distributed load acting upon the fictitious beam in the two segments is resolved into simpler figures. In this manner, the resultant forces are calculated easier:



$$\overline{R_1} = \frac{30.6}{2} = 90\,kNm^2;\; f = \frac{ql^2}{2}\frac{I_1}{I_1} = \frac{20.6^2}{8}.1 = 90\,kNm;\; \overline{R_2} = \frac{2}{3}.fl = \frac{2}{3}.90.6 = 360\,kNm^2;$$

$$\overline{R_3} = \frac{105{,}650.2{,}1}{2} = 110{,}932\,kNm^2;\; \overline{R_4} = \frac{138{,}402.2{,}1}{2} = 145{,}322\,kNm^2.$$



Next step in the solution is the support reactions determination.

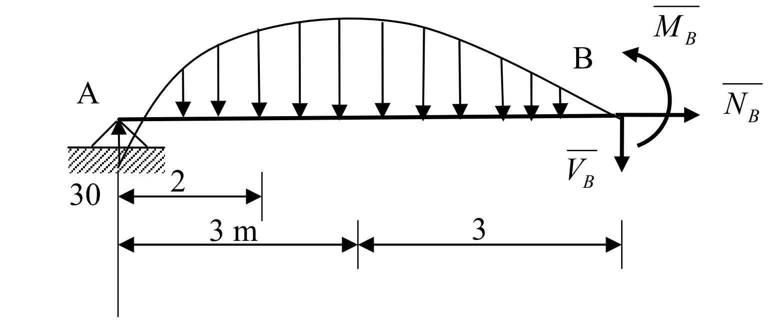
$\sum H = 0$; $A_H = 0$;

$\sum M_A = 0$; $90.2 - 360.3 - 110{,}932.6{,}7 + 145{,}322.7{,}4 + 6\overline{B} = 0$; $6\overline{B} = 567{,}862$; $\overline{B} = 94{,}644\,kNm^2$;

$\sum M_B = 0$; $-90.4 + 360.3 - 110{,}932.0{,}7 + 145{,}322.1{,}4 - 6\overline{A_V} = 0$; $6\overline{A_V} = 845{,}798$; $\overline{A_V} = 140{,}966\,kNm^2$

- Check: $\sum V = 0$; $140{,}966 + 90 - 360 + 94{,}644 - 110{,}932 + 145{,}322 = 0$; $470{,}932 - 470{,}932 = 0$.

According to the *Mohr*'s analogy method $EI_1 w_B = \overline{M_B}$. This means that the bending moment in the section B of the fictitious beam has to be found first. The well-known method of section is used.



$\sum M = 0$;

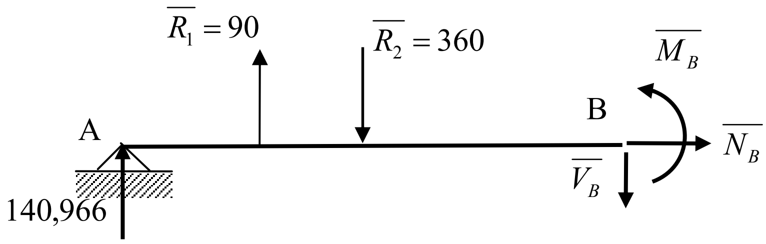
$\overline{M_B} - 140{,}966.6 - 90.4 + 360.3 = 0$;

$\overline{M_B} = 125{,}796\ kNm^3$.

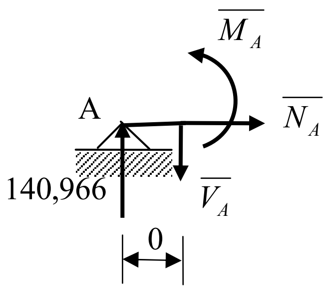
$EI_1 w_B = 125{,}796\ kNm^3$.

$EI_1 = 2.10^4.11620 = 2{,}324.10^8\ kNcm^3$;

$$w_B = \frac{125{,}796.10^6}{2{,}324.10^8} = 0{,}541\ cm.$$



The slope α of the section A must be determined, too. Considering the relation $EI_1\alpha_A = \overline{V_A}$, the conclusion that the shearing force $\overline{V_A}$ must be found first, is made.



The method of section is applied again:

$\sum V = 0$; $140{,}966 - \overline{V_A} = 0$; $\overline{V_A} = 140{,}966\ kNm^2$.

Then: $EI_1\alpha_A = 140{,}966\ kNm^2$.

Finally:

$$\alpha_A = \frac{140{,}966.10^4}{2{,}324.10^8} = 0{,}00607\ rad = 0{,}00607.\frac{180^0}{\pi} = 0{,}348^0.$$

7.5. STATICALLY INDETERMINATE BEAMS SUBJECTED TO BENDING

In the case of the *externally* statically indeterminate beams the number of the statical unknowns (*support reactions*) is bigger than the number of the equilibrium equations which can be used. Thus, if the elastic line of the beam has to be obtained not only the *kinematical* initial parameters exist but also the unknown *statical*. Then, the *kinematical* as well as the *dynamical* boundary conditions must be written for their determination. The *dynamical* boundary conditions include the bending moments and the shearing forces in the definite beam section.

Problem 7.5.1: The beam given has a length l and stiffness EI and it is acted upon by a single force F, as shown. Determine the EI-multiple value of the deflection's functions $w(x)$ in the two segments.

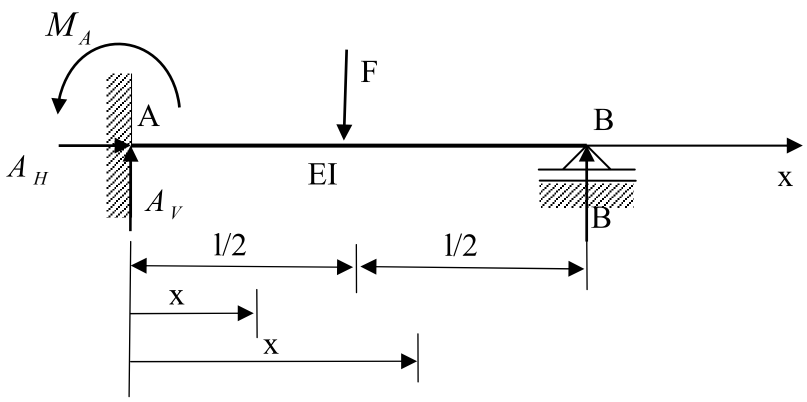


Fig.7.14: Statically indeterminate beam subjected to bending

The beam given contains 4 support reactions while the number of the equilibrium equations is 3. Then, the beam has 4-3 = 1 degree of the statical indeterminacy.

The force F divides the beam into two segments. The distance x is measured from *the left beam end* for the both segments.

The *mixed* kinematical and dynamical boundary conditions are written:

1) $w_1(0)=0$; 2) $\alpha_1(0)=0$; 3) $w_2(l)=0$; 4) $M_2(l)=0$.

Further, the *universal* equation of the elastic line is written for the two segments:

$$EIw_1(x)=EIw_0+EI\alpha_0 x+\frac{M_A}{2!}(x-0)^2-\frac{A_V}{3!}(x-0)^3;$$

$$EIw_2(x)=EIw_0+EI\alpha_0 x+\frac{M_A}{2!}(x-0)^2-\frac{A_V}{3!}(x-0)^3+\frac{F}{3!}\left(x-\frac{l}{2}\right)^3.$$

To justify the boundary condition 2) the following differentiation is made:

$$EI\alpha_1(x)=EIw'_1(x)=EI\alpha_0+M_A x-\frac{A_V x^2}{2}.$$

To justify the boundary condition 4) the expression for the bending moment M_2 must be found – the differential equation of the elastic line for the second segment is written:

$$EIw''_2(x)=-M_2(x),$$

and it is obtained:

$$M_2(x)=A_V x-M_A-F\left(x-\frac{l}{2}\right).$$

Then, the kinematical boundary conditions 1) and 2) are justified:

1) $EIw_1(0)=EIw_0+EI\alpha_0.0+\frac{M_A}{2!}.0^2-\frac{A_V}{3!}.0^3=0$;

2) $EI\alpha_1(0)=EI\alpha_0+M_A.0-\frac{A_V.0^2}{2}=0$,

and it is carried out $EIw_0=0$, $EI\alpha_0=0$.

Thus, the equation of $EIw_2(x)$ becomes:

$$EIw_2(x)=\frac{M_A}{2!}x^2-\frac{A_V}{3!}x^3+\frac{F}{3!}\left(x-\frac{l}{2}\right)^3.$$

Further, the conditions 3) and 4) are written:

3) $EIw_2(l)=\frac{M_A}{2!}l^2-\frac{A_V}{3!}l^3+\frac{F}{3!}\left(l-\frac{l}{2}\right)^3=0$;

4) $M_2(l)=A_V l-M_A-F\left(l-\frac{l}{2}\right)=0$,

and the support reactions are calculated $A_V=\frac{11F}{16}, M_A=\frac{3Fl}{16}$.

Finally, the functions defining the elastic line are obtained:

$$EIw_1(x)=-\frac{11F}{96}x^3+\frac{3Fl}{32}x^2;$$

$$EIw_2(x)=-\frac{11F}{96}x^3+\frac{F}{3!}\left(x-\frac{l}{2}\right)^3+\frac{3Fl}{32}x^2.$$

If the diagrams of the bending moment $M(x)$ and the shearing force $V(x)$ have to be built, then the differential equations $EIw''(x)=-M(x)$ and $M'(x)=V(x)$ will be applied.

7.6. THE INFLUENCE OF THE SHEARING FORCE ON THE ELASTIC LINE DIFFERENTIAL EQUATION

The elastic line differential equation $EI_y w''(x) = -M_y(x)$ is valid in the case of *pure bending* only. If the shearing force $V_z(x)$ exists in the beam, then, this equation will change. To derive the new relation of the beam elastic line the principle of superposition must be applied.

The deformation of the infinitesimal beam segment of length dx subjected to the shearing forces V_z only is considered. This is the well-known loading conditions when the shearing forces' influence must be found. In such case the shearing forces V_z will cause the displacements of the beam sections in two parallel planes where the right situated section will move downward with respect to the left situated section.

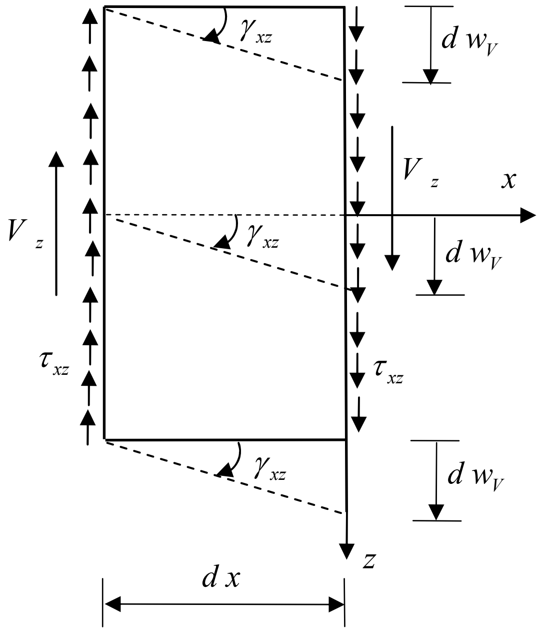


Fig. 7.15: A beam element subjected to pure shearing

In the case of the beam subjected to pure shearing all of the points of the right situated section have equal vertical displacements dw_V after the deformation. These vertical displacements are very small and they are supposed to be arcs. Their magnitudes are:

$$dw_V = \gamma_{xz}\, dx . \tag{7.54}$$

According to the Hook's law the relation between shearing stresses and strains is

$$\tau_{xz} = G\gamma_{xz} , \tag{7.55}$$

where G is the shear modulus.

In the other side, the shearing stresses caused by V_z can be represented as

$$\tau_{xz} = k\frac{V_z}{A} , \tag{7.56}$$

where k is the shear coefficient and A is the area of the cross-section.

After equalizing of the right-hand sides of (7.55) and (7.56) it is obtained for shearing strains:

$$\gamma_{xz} = k\frac{V_z}{GA} . \tag{7.57}$$

Then, the vertical displacement dw_V is expressed by shearing force V_z in the manner

$$dw_V = k\frac{V_z}{GA} dx . \tag{7.58}$$

Thus, it is determined $w'_V = \frac{d\,w_V}{dx} = k\frac{V_z}{GA}$. After differentiation it is obtained

$$w''_V = \frac{d\,w'_V}{dx} = \frac{k}{GA}.\frac{d\,V_z}{dx}. \qquad (7.59)$$

Further, using the differential equation $\frac{d\,V_z}{dx} = -q$, where q is the intensity of the distributed load, it is found

$$w''_V = -k\frac{q}{GA}. \qquad (7.60)$$

If in some problem the influence of the shearing force must be determined only, then, the differential equation $w'_V = k\frac{V_z}{GA}$ will be integrated. It will be obtained $w_V = \frac{k}{GA}\int V_z\,dx + C_1$, where C_1 is a constant which will be found applying the *kinematical* boundary condition.

Then, using the differential relation $\frac{dM_y}{dx} = V_z$, it is obtained

$$w_V(x) = \frac{k}{GA}M_y(x) + C_1. \qquad (7.61)$$

Formula (7.61) shows that the elastic line caused by the shearing force V_z only has the same shape like the bending moment diagram.

In the case of a beam subjected to *pure bending* the differential equation of the elastic line is:

$$w'' = -\frac{M_y}{EI_y}.$$

The shearing force is taken into account by the principle of superposition, namely by adding of the expression $-k\frac{q}{GA}$ in the right-hand side in the formula above.

Thus, the elastic line differential equation in the case of the beam subjected to bending combined with shear is:

$$w'' = -\frac{M_y}{EI_y} - k\frac{q}{GA}. \qquad (7.62)$$

Further, the function of the vertical displacements is obtained by integration.

The shear coefficient k taking part in expression (7.62) is determined in Chapter 11:

$$k = \frac{A}{I_y^2}\iint_{(A)}\frac{S_y^2(z)}{b^2(z)}dA. \qquad (7.63)$$

It is obvious that only geometrical characteristics of the cross-section are used, as follow: A - the cross-sectional area; I_y - the moment of inertia about y-axis; $b(z)$ - the width of the cross-section in the random level; $S_y(z)$ - the statical moment about y-axis of the portion of the cross-section under or above the level considered. The values of the shear coefficient for common geometrical shapes are:

- $k = 1{,}2$ for rectangular cross-sections;
- $k = 10/9$ for solid circular cross-sections;
- $k = 2$ for *high* I-profiles cross-section and $k = 2{,}4$ for *low* I-profiles cross-section.

Problem 7.6.1: Find the midsection's deflection of the beam shown in the fig. 7.4, if the influence of the shearing force is taken into account. The beam has rectangular cross-section and the *Poisson*'s coefficient is $\nu = 0{,}25$.

To find the midsection's deflection the principle of superposition has to be applied where the influences of the bending moment M_y and the shearing force V_z must be taken separately:

$$w(x) = w_{My}(x) + w_V(x).$$

Using the function (7.22) it is determined $w_{My}\left(\frac{l}{2}\right) = \frac{5ql^4}{384\ EI_y}$.

To take into account the influence of the shearing force the expression (7.61) is used where the integration constant $C_1 = 0$ is obtained by the kinematical boundary condition $w(0) = 0$. Then, it is determined:

$$w_V = \frac{kql^2}{8GA}.$$

After that:

$$w\left(\frac{l}{2}\right) = \frac{5ql^4}{384EI_y} + \frac{kql^2}{8GA}.$$

The expression above can be represented in the manner:

$$w\left(\frac{l}{2}\right) = \frac{5ql^4}{384EI_y}\left(1 + \frac{48}{5}k\frac{i_y^2}{l^2}\frac{E}{G}\right), \text{ където } i_y^2 = \frac{I_y}{A} = \frac{bh^3/12}{bh} = \frac{h^2}{12}.$$

Further, applying the relation $G = \frac{E}{2(1+\nu)}$ it is found $\frac{E}{G} = 2(1+\nu) = 2(1+0{,}25) = 2{,}5$.

Finally:

$$w\left(\frac{l}{2}\right) = \frac{5ql^4}{384EI_y}\left(1 + \frac{2{,}4}{\left(\frac{l}{h}\right)^2}\right).$$

It is obvious, if the ratio $\frac{l}{h}$ is equal to 10, then the influence of the shearing force on the vertical displacement will be equal to 2,4%.

7.7. DETERMINATION OF THE ELASTIC LINE EQUATION IN THE CASE OF A BEAM SUBJECTED TO DOUBLE BENDING

The elastic line determination in the case of a beam subjected to double bending is more complicated than this one in the case of special bending.

Here, the *Bernoulli*'s hypothesis says that each beam section has rotated about the neutral axis and the random longitudinal beam fibre belongs to the plane normal to the neutral axis after deformation.

In the case of a beam subjected to double bending the bending moments about the principal axis y and z are different than zero. The normal stresses caused by these moments are

$$\sigma_x = \frac{M_y}{I_y}z - \frac{M_z}{I_z}y, \tag{7.64}$$

and the equation of the neutral axis n is

$$z = \frac{M_z}{M_y}\frac{I_y}{I_z}y. \tag{7.65}$$

It is seen from (7.65) that the neutral axis in the case of a beam subjected to double bending does not coincide to the principal axis y and it changes its position with respect to the principal beam axis y and z for every beam section. Then, the random longitudinal beam fibre will deform in different way for different beam section. It is following from this that the beam axis after deformation will be the spatial curve and for its determination the deflections' functions $w(x)$ and $v(x)$ must be found separately applying the principle of superposition. Here, $v(x)$ is the function of the deflections along the principal axis y.

First, the forces acting in the plane xz are considered – they cause the bending moment M_y. Then, to determine the deflection's function $w(x)$ the familiar differential equation (7.6) has to be integrated.

After that, the loads acting in the plane xy are investigated. They cause the bending moment M_z and give the deflections $v(x)$ along the y-axis – the differential equation of $v(x)$ can be derived. For that purpose, the elastic line's curvature due to the bending moment M_z only is considered. To obtain the correct sign in the equation's right-hand side the two sketches of the beam axis deformation are drawn, as follow:

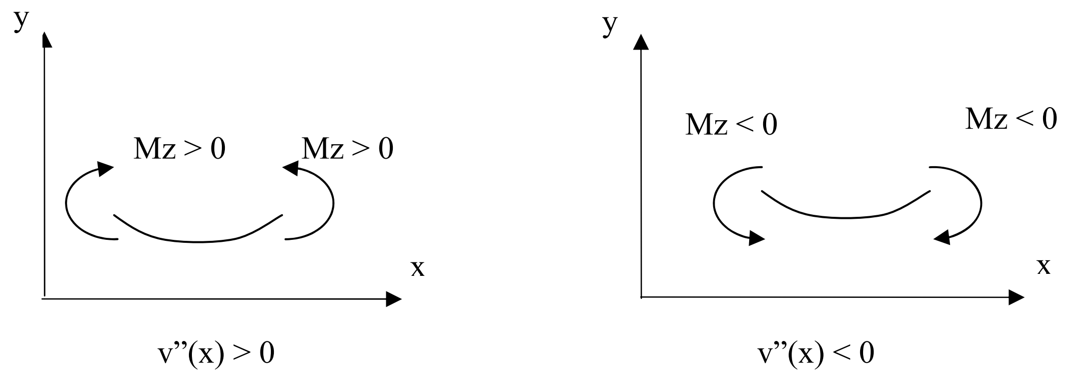


Fig.7.16: The beam elements subjected to M_z after deformation

It is obvious that v'' and M_z have the same sign. Then, the differential equation of $v(x)$ has the form:

$$v'' = +\frac{M_z}{EI_z}. \qquad (7.66)$$

Next step is the integration of this differential equation.

Finally, the method of superposition must be applied. If the displacement d_M of the beam section M at a distance x_M from the beam left end must be determined, then the deflections $w(x_M)$ and $v(x_M)$ have to be found applying expressions (7.6) and (7.66), respectively. Using the fact, that they are perpendicular to each other, it will be obtained $d_M = \sqrt{w^2(x_M) + v^2(x_M)}$.

CHAPTER 8

MOMENTS OF INERTIA

8.1. DEFINITION

8.1.1. MOMENTS OF INERTIA ABOUT AXES

A plane figure representing cross-section of a beam is considered. The figure contains an infinite number of elements of area dA, as shown in fig.8.1. Then, the total area of the cross-section will be

$$A = \iint_{(A)} dA . \tag{8.1}$$

Area is the simplest geometrical characteristic of the cross-section and it has dimension ***length*^{2}**. The area is always positive and does not depend on the coordinate system chosen.

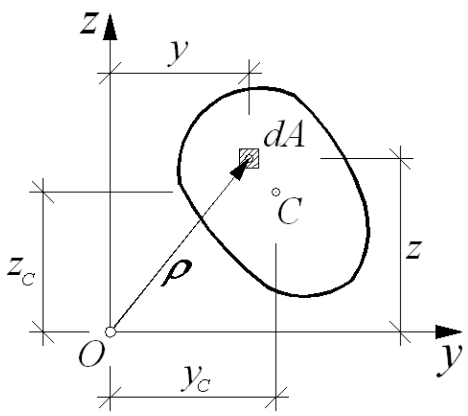


Fig. 8.1: Cross-section of a beam

The moments of inertia about axes y and z, respectively, are defined by integrals

$$I_y = \iint_{(A)} z^2 dA ;$$

$$I_z = \iint_{(A)} y^2 dA . \tag{8.2}$$

8.1.2. POLAR MOMENT OF INERTIA

Polar moment of inertia or moment of inertia about a point (pole) is

$$I_O = \iint_{(A)} \rho^2 dA , \tag{8.3}$$

where ρ is the distance from the area dA to the pole – point O.

If the pole about which the polar moment of inertia must be calculated is the origin of the coordinate system, then, $\rho^2 = y^2 + z^2$, and

$$I_O = \iint_{(A)} \rho^2 dA = \iint_{(A)} y^2 dA + \iint_{(A)} z^2 dA . \tag{8.4}$$

Therefore

$$I_O = I_y + I_z , \tag{8.5}$$

i.e., the sum of the moments of inertia about two axes perpendicular to each other and passing through a given point is equal to the polar moment of inertia about the same point.

8.1.3. PRODUCT OF INERTIA

Product of inertia of the figure about axes y and z is

$$I_{yz} = \iint_{(A)} yz\,dA . \tag{8.6}$$

If the cross-section has axis of symmetry, then the product of inertia about that axis will be equal to zero.

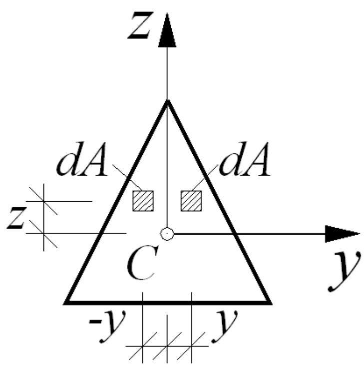


Fig. 8.2: Cross-section possessing the axis of symmetry

It can be seen in fig. 8.2 when the figure has the axis of symmetry it always can be separated into two parts having similar coordinates. Besides, the difference between the coordinates is in the sign only: $z_1 = z_2 = z$ and $y_1 = y$ $y_2 = -y$. Then, calculating the product of inertia it is obtained:

$$I_{yz} = \iint_{(A)} yz\,dA = \iint_{\left(\frac{A}{2}\right)} yz\,dA + \iint_{\left(\frac{A}{2}\right)} (-y) z\,dA = 0 . \tag{8.7}$$

The moments of inertia have dimension ***length***^{4}.

The moments of inertia about axes and the polar moment of inertia are always positive while the product of inertia can be positive, negative or even zero.

8.2. MOMENTS OF INERTIA OF THE RECTANGLE

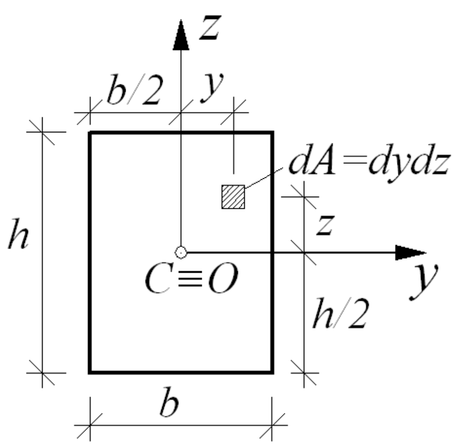


Fig. 8.3: Rectangle

Here, the infinitesimal area is

$$dA = dydz \,. \tag{8.8}$$

Then, joining the first of expressions (8.2) and expression (8.8) and performing the following transformations it is carried out:

$$I_y = \iint\limits_{(A)} z^2 dA = \iint\limits_{(A)} z^2 dydz = \int\limits_{-\frac{b}{2}}^{\frac{b}{2}} dy \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz = \left(\frac{b}{2} + \frac{b}{2}\right)\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{bh^3}{12}; \tag{8.9}$$

Further, repeating the same action about the second expression of (8.2) and expression (8.8) it is obtained:

$$I_z = \frac{hb^3}{12} \,. \tag{8.10}$$

The rectangle has two axes of symmetry and because of that:

$$I_{yz} = 0 \,. \tag{8.11}$$

Finally, the expressions (8.9) and (8.10) are substituted in (8.5) and it is found:

$$I_O = \frac{bh^3}{12} + \frac{hb^3}{12} = \frac{bh}{12}\left(b^2 + h^2\right) = \frac{A}{12}\left(b^2 + h^2\right). \tag{8.12}$$

8.3. THE PARALLEL AXES THEOREM (STEINER'S THEOREM)

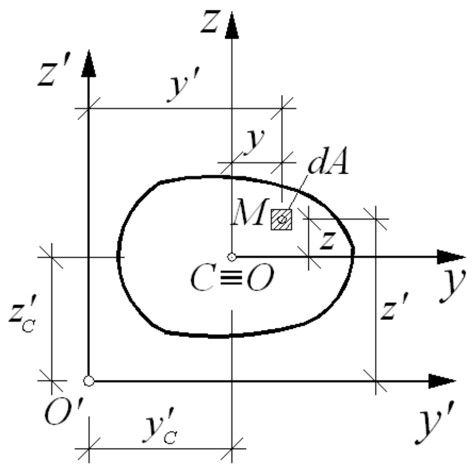


Fig. 8.4: Cross-section of a beam with two coordinate systems parallel to each other

Let y and z to be the central axes about which the moments of inertia I_y, I_z and the product of inertia I_{yz} to be known. Let the axes y' and z' to be parallel to the axes y and z, respectively.

The task is the moments of inertia $I_{y'}$, $I_{z'}$ and the product of inertia $I_{y'z'}$ to be found.

Solution:

Let dA to be an infinitesimal area in the vicinity of point M of coordinates y and z.

Then, applying the first formula of (8.2) it is obtained:

$$I_{y'} = \iint\limits_{(A)} z'^2 dA \,. \tag{8.13}$$

Further, investigating fig. 8.4 it is evident

$$z' = z_C' + z \,. \tag{8.14}$$

The expression (8.14) is substituted in (8.13) and the transformations are made, as follow:

$$I_{y'} = \iint\limits_{(A)} (z + z_C')^2 dA = \iint\limits_{(A)} z^2 dA + 2z_C' \iint\limits_{(A)} z dA + z_C'^2 \iint\limits_{(A)} dA . \qquad (8.15)$$

After that, using (8.2) it can be seen that the first integral in the right-hand side is the moment of inertia $I_{y'}$ while applying (8.1) it can be determined that the third integral is the figure's area A.

The coordinate of the figure's center of gravity is found by the well-known formula of Theoretical mechanics:

$$z_C = \frac{\iint\limits_{(A)} z dA}{A} , \qquad (8.16)$$

where $S_y = \iint\limits_{(A)} z dA$ is the ***statical moment about axis*** y.

But $z_C = 0$ and $\iint\limits_{(A)} z dA = 0$.

Then, the expression (8.15) will be

$$I_{y'} = I_y + A z_C'^2 . \qquad (8.17)$$

Similarly, the following relations are derived

$$I_{z'} = I_z + A y_C'^2 . \qquad (8.18)$$

$$I_{yz'} = I_{yz} + A y_C' z_C' . \qquad (8.19)$$

8.4. RELATIONS BETWEEN THE MOMENTS OF INERTIA ABOUT AXES ROTATED TO EACH OTHER

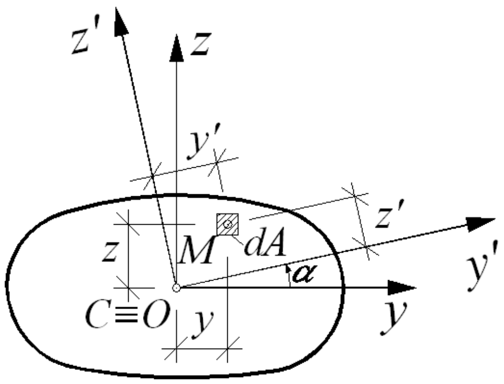


Fig. 8.5: Cross-section of a beam with two coordinate systems rotated to each other at an angle α

Let y and z to be the central axes about which the moments of inertia I_y, I_z and the product of inertia I_{yz} to be known. Let the axes y' and z' to be rotated at an angle α with respect to the axes y and z. Now, the aim is the moments of inertia $I_{y'}, I_{z'}$ and the product of inertia $I_{y'z'}$ to be obtained.

Solution:

The relations between the coordinates of a point M with respect to two coordinate systems are:

$$y' = y\cos\alpha + z\sin\alpha , \; z' = -y\sin\alpha + z\cos\alpha . \qquad (8.20)$$

First, the moment of inertia $I_{y'}$ will be represented in form (8.13). Then, the second expression of (8.20) will be substituted in (8.13):

$$I_{y'} = \iint_{(A)} (-y\sin\alpha + z\cos\alpha)^2 dA = \sin^2\alpha \iint_{(A)} y^2 dA - 2\sin\alpha\cos\alpha \iint_{(A)} yz dA + \cos^2\alpha \iint_{(A)} z^2 dA . \quad (8.21)$$

Further, the trigonometric relations are used:

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}; \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}; \quad \sin 2\alpha = 2\sin\alpha\cos\alpha . \quad (8.22)$$

Formulas (8.2) and (8.6) are replaced in (8.21) and it is obtained:

$$I_{y'} = \frac{1-\cos 2\alpha}{2} I_z - \sin 2\alpha I_{yz} + \frac{1+\cos 2\alpha}{2} I_y . \quad (8.23)$$

The final form of expression for moment of inertia $I_{y'}$ is

$$I_{y'} = \frac{1}{2}(I_y + I_z) + \frac{1}{2}(I_y - I_z)\cos 2\alpha - I_{yz}\sin 2\alpha . \quad (8.24)$$

The moment of inertia $I_{z'}$ and the product of inertia $I_{y'z'}$ are obtained in similar way:

$$I_{z'} = \frac{1}{2}(I_y + I_z) + \frac{1}{2}(I_z - I_y)\cos 2\alpha + I_{yz}\sin 2\alpha . \quad (8.25)$$

$$I_{y'z''} = -\frac{1}{2}(I_y - I_z)\sin 2\alpha + I_{yz}\sin 2\alpha . \quad (8.26)$$

As a conclusion, it can be said that the sum of the moments of inertia about two axes perpendicular to each other remains constant:

$$I_{y'} + I_{z'} = I_y + I_z = I_O . \quad (8.27)$$

8.5. PRINCIPAL MOMENTS AND PRINCIPAL AXES OF INERTIA

The moments of inertia about ***principal axes*** have extreme values relative to the all moments of inertia. Besides, the product of inertia about the same axes is equal to zero.

The moments of inertia about the principal axes are called ***principal moments of inertia***. They can be determined by the formula:

$$I_{1,2} = I_{\max,\min} = \frac{I_y + I_z}{2} + \sqrt{\left(\frac{I_y + I_z}{2}\right)^2 + I_{yz}{}^2} . \quad (8.28)$$

Angles α_1 and α_2 which the principal axes of inertia make with the horizontal axis y can be found by expression:

$$tg\alpha = \frac{2I_{yz}}{I_y - I_z}, \quad (8.29)$$

where the relation $\alpha_2 = \alpha_1 + \frac{\pi}{2}$ exists.

CHAPTER 9

TORSION

9.1. DEFINITION

Torsion is a type of deformation in which the transverse sections of a beam twist relative to each other under the action of **external torsion moments only. The external forces situated normally to the beam axis cause the torsion moments because they do not intersect the axis. The torsion moments' planes of action are perpendicular to the longitudinal beam axis.**

The pure torsion conditions in the curved beams might be caused by different loading configuration than mentioned above. As an example, in the thin-walled beams the torsion can arise if the point of application of the transverse force does not coincide with *the cross-section's shear center (center of twist)*; in this case, the torsion is combined with bending. However, if the bending moment is very smaller with respect to the torsion one, then the case of a pure torsion has to be investigated.

In practice, the shafts, the coil springs and other machines are predicted to work in pure torsion conditions.

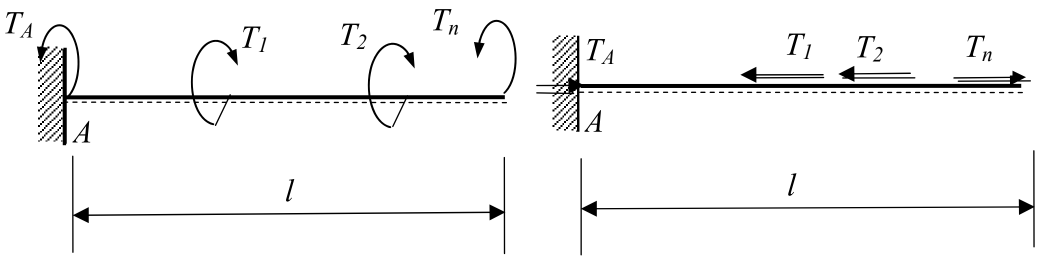


Fig. 9.1: Beam working in pure torsion condition

Pure torsion obeys the condition at any beam section only *the torsion moment* T to be different than zero, *i.e.*

$$T \neq 0; \qquad N = V_y = V_z = M_y = M_z = 0. \tag{9.1}$$

9.2. THE TORSION MOMENT DIAGRAM

The beam shown in fig.9.1 is considered and *the method of section* is applied. The beam part loaded by a small number of external moments is chosen for investigation and the positive sense of the torsion moment is introduced, as follow: if the torsion moment rotates in *the counterclockwise direction* when we look at the section cut, then its sense is *positive* and vice versa.

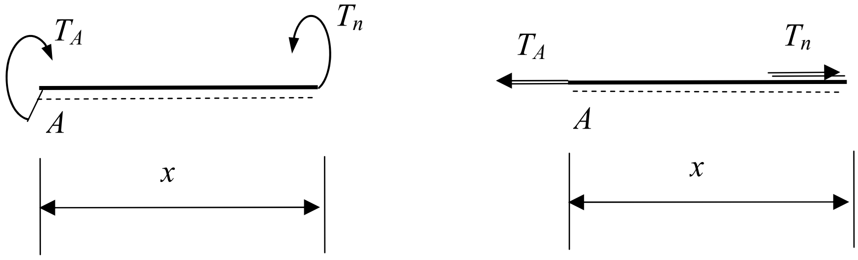


Fig. 9.2: The torsion moment in beam section $n-n$

The moment equilibrium equation about the longitudinal beam axis for the beam part chosen must be written. In this way, it can be concluded that **the torsion moment is equal as a magnitude and opposite as a sense to the external moments acting upon the beam part considered.**

- Build the torsion moment diagrams of the beams shown in Problems 9.1 and 9.2.

Problem 9.1

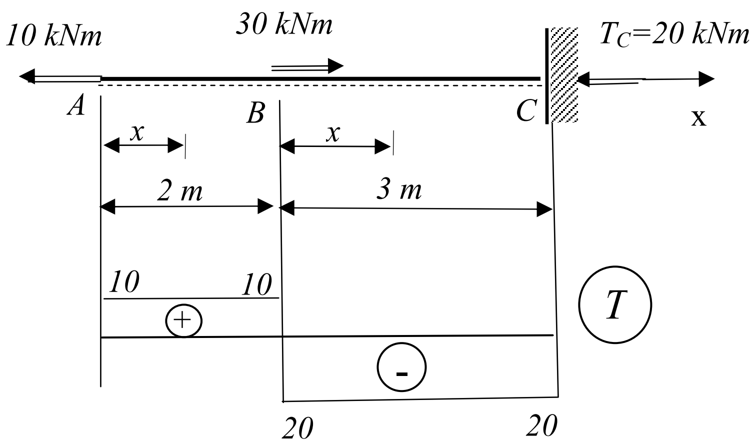


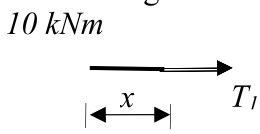
Fig. 9.3: A beam working in pure torsion conditions

Because of the loading, only the moment M_C situated in the fixed support, as shown, is the unknown support reaction and it will be determined by the condition:

$$\sum M_x = 0; -T_C + 30 \quad 10 = 0; T_C = 20\,kNm.$$

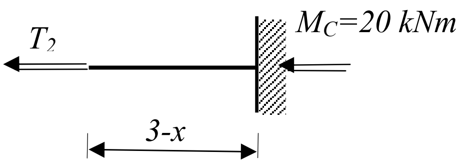
The beam contains two segments. They will be considered separately and the functions of the torsion moments will be obtained, as follow:

I segment: $0 \le x \le 2\,\text{m}$



$$\sum M_x = 0; T_1 - 10 = 0; T_1 = 10\,kNm.$$

I I segment: $0 \le x \le 3\text{m}$



$$\sum M_x = 0; -20 - T_2 = 0; T_2 = -20 kNm.$$

- Torsion moment diagram

Note: The positive values of the torsion moment diagram have to be drawn above the zero line!

Problem 9.2

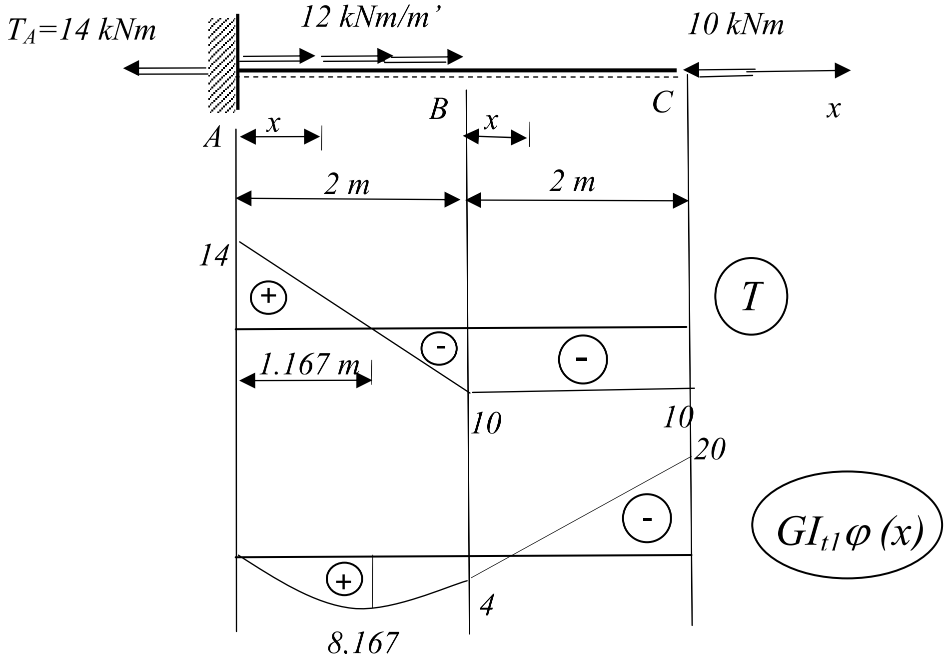


Fig. 9.4: A beam working in pure torsion conditions

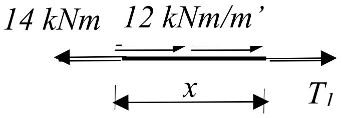
The uniformly distributed torsion moment of intensity $12\,kN\,m/m'$ is applied in the segment *AB*. It will be substituted for a resultant torsion moment of magnitude equal to the product between the distributed moment intensity and the length of the segment.

After that, to find the moment support reaction, the relevant equilibrium equation will be written:

$\sum M_x = 0;\ -T_A + 12.2 - 10 = 0;\ T_A = 14\,kNm$.

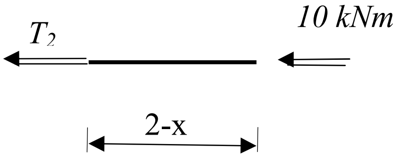
Next step in the solution is the torsion moment functions determination:

I segment: $0 \le x \le 2\,\mathrm{m}$



$\sum M_x = 0;\ T_1 + 12\mathrm{x} - 14 = 0;\ T_1 = -12\mathrm{x} + 14;\ \ \mathrm{T_1}(0) = 14\,\mathrm{kNm};\ T_1(2) = -12.2 + 14 = -10\,\mathrm{kNm}$

I I segment: $0 \le x \le 2\,\mathrm{m}$



$\sum M_x = 0;\ -18 - T_2 = 0;\ T_2 = -10\,kNm$.

- Torsion moment diagram

9.3. BEAMS OF SOLID CIRCULAR AND HOLLOW CIRCULAR CROSS-SECTIONS

The type of the cross-section influences too much on the stresses and strains in beams subjected to a pure torsion.

9.3.1. STRESSES

The experiment with the rectilinear beams of solid ***circular*** cross-sections has been made, as follow: The web of lines is drawn on the beam surface. Some of the lines are rectilinear and parallel to the longitudinal axis while the others are circles lying in the planes which are normal to the beam axis. Thus, the rectangles on the cylindrical surface are obtained. Every cross-section has definite points onto the circle through which the radial lines are built.

Then, the beam is loaded so that *the pure torsion* conditions to be performed and it is concluded after deformation:

- All of the lines parallel to the beam axis have rotated at the same angle γ with respect to their initial positions. Besides, the rectangles onto the cylindrical surface have become parallelograms;

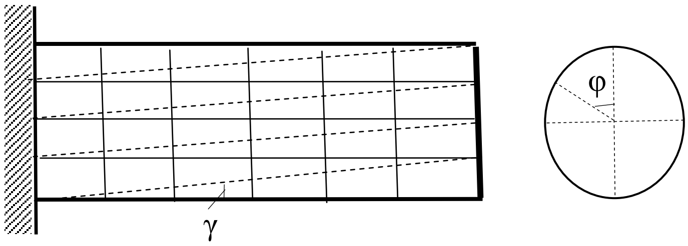


Fig. 9.5: Deformation of a beam working in pure torsion conditions

- The beam cross-sections remains plane, circular, and at the same distances relative to each other as in the beginning;
- Any beam section has been rotated at an angle φ, named ***angle of twist*** relative to its initial position, i.e. the section rotates with respect to the beam axis as the rigid plate do;
- The radial lines remains rectilinear and the lengths of the radii do not change.

On the basis of the experiment, it can be concluded that the Bernoulli's ***hypothesis*** takes place, namely the plane sections before deformation remain plane after deformation. ***Besides, since the torsion moments only act in the cross-sectional planes, the shearing stresses arise there, while the normal stresses are equal to zero. Furthermore, according to the theorem of the shearing stresses equivalence it is proved that the shearing stresses in the longitudinal beam sections are equal to these ones in the cross-section.***

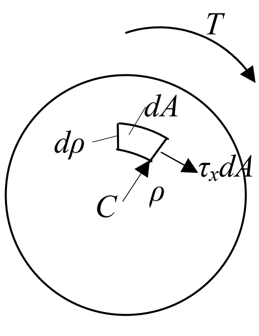


Fig. 9.6:
Cross-section of a beam working in pure torsion condition

Then, to determine the stresses, the beam working in pure torsion conditions will be examined, as follow: The beam section at distance x from the left end is given in Fig. 9.6. The torsion moment in the section is labelled by T and it is *the internal force different than zero only*. T must be represented as a sum of the moments about the cross-section's center of gravity C of the forces perpendicular to the radii passing through their points of application. The forces like these are directly related to the shearing stresses.

Thus, the stress conditions in a beam working in pure torsion are similar to these ones in a beam working in ***pure shear***, i.e. in the both cases the ***shearing stresses only*** exist.

Further, infinitesimal force $\tau_x dA$ acting upon infinitesimal plane dA will be considered (Fig.9.6). The moment of the force about the beam axis is $(\tau_x dA)\rho$. Then, the torsion moment in the cross-section T is:

$$T = \iint_{(A)} \rho \tau_x dA \,. \qquad (9.2)$$

The integration will be made when the law of shearing stresses distribution in the cross-sectional plane is obtained, as follow:

Two cuts through the beam separate a portion of length dx and thickness $d\rho$. The left section of the portion has to be supposed fixed. Then, under the action of the torsion moments the right section will rotate relative to the left one at an angle $d\varphi$, while every generant will rotate at an angle γ with respect to its initial position. Angle γ is ***the angle of relative torsion***. The positions of infinitesimal beam portion, generant DB and radius CB before and after deformation are given in Fig. 9.7.

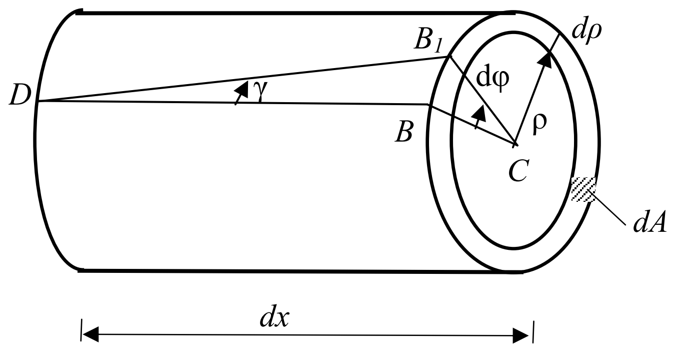


Fig. 9.7: Infinitesimal portion of a beam working in pure torsion

The length of the arc $\overline{BB_1}$ can be represented in two manners:

$$\overline{BB_1} = (\rho + d\rho)d\varphi = \rho\, d\varphi \qquad (9.3)$$

$$\overline{BB_1} = \gamma_x dx \,. \qquad (9.4)$$

The shear strain is determined by comparing of the two right-hand sides:

$$\gamma_x = \rho \frac{d\varphi}{dx} \,. \qquad (9.5)$$

According to the Hook's law:

$$\tau_x = G\gamma_x \,. \qquad (9.6)$$

The shearing strain γ_x from (9.5) is substituted in (9.6) and the shearing stresses are:

$$\tau_x = G\rho \frac{d\varphi}{dx} \,. \qquad (9.7)$$

Further, expression of τ_x is put in (9.2):

$$T = \iint_{(A)} \rho G \rho \frac{d\varphi}{dx} dA = G \frac{d\varphi}{dx} \iint_{(A)} \rho^2 dA \,. \qquad (9.8)$$

$I_t = \iint_{(A)} \rho^2 dA$ is the cross-sectional polar moment of inertia.

Then, (9.8) becomes:

$$\frac{d\varphi}{dx} = \frac{T}{GI_t}. \tag{9.9}$$

Finally, taking $\frac{d\varphi}{dx}$ from (9.9) and putting it in (9.7), for τ_x is obtained:

$$\tau_x = \frac{T}{I_t}\rho. \tag{9.10}$$

It is obvious the shearing stresses function is linear with respect to the distance from the cross-section's center of gravity to the random cross-section's point.

The shearing stresses in a beam working in pure torsion are directly proportional to the distance from the cross-section's center of gravity to the point considered. When $\rho = 0$***, from (9.10) follows*** $\tau_x = 0$***. The shearing stresses have the biggest values when*** $\rho = R$***:***

$$\tau_{\max} = \frac{T}{I_t}R. \tag{9.11}$$

The shearing stresses distribution of the solid circular and the hollow circular cross-sections are shown in Fig. 9.8.

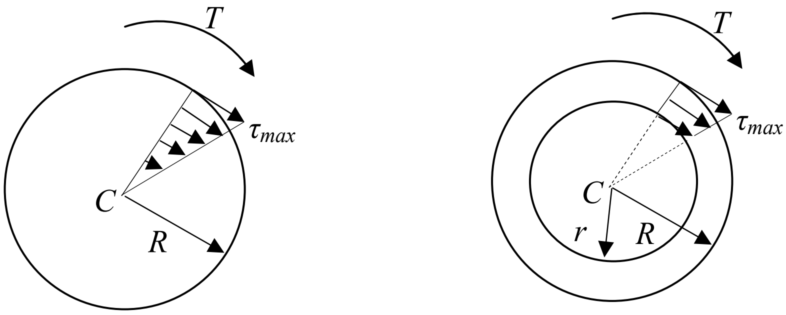


Fig. 9.8: Shearing stresses distribution

The ratio between the cross-section's polar moment of inertia and the radius, labeled by W_t, ***is named section modulus of a beam working in pure torsion conditions:***

$$W_t = \frac{I_t}{R} \tag{9.12}$$

Then, the maximum shearing stresses in the cross-section are:

$$\tau_{\max} = \frac{T}{W_t}. \tag{9.13}$$

It can be said as a conclusion, the experiments show that all of the expressions derived for solid circular members working in pure torsion can be applied in the case of the hollow circular cross-sections.

9.3.2. DETERMINATION OF THE POLAR MOMENT OF INERTIA I_t AND SECTION MODULUS W_t

The solid circular cross-section of radius R is considered where the part of thickness $\mathrm{d}\rho$ is detached from the beam (Fig.9.6). This part must be divided into the infinitesimal segments of areas $\mathrm{d}A$. All of them are situated at a distance ρ from the cross-section's center of gravity *C*. Thus the polar moment of inertia of the part detached is:

$$\overline{I_t} = \iint_a \rho^2 dA = \rho^2 \iint_A dA = \rho^2 A \tag{9.14}$$

The area of the part considered can be determined by the expression:

$$A = 2\pi\rho d\rho \,. \tag{9.15}$$

Then, after substitution of (9.15) in (9.14), it is obtained

$$\overline{I_t} = 2\pi\rho^3 d\rho \,. \tag{9.16}$$

Further, to determine the polar moment of inertia of the entire figure, the integration will be made, as follow:

$$I_t = 2\pi\int_0^R \rho^3 d\rho = 2\pi\left(\rho^4/4\right)\Big|_0^R = \frac{\pi R^4}{2} \,. \tag{9.17}$$

The expression of the polar moment of inertia with respect to the diameter D of the solid circular cross-section is:

$$I_t = \frac{\pi D^4}{2} \,. \tag{9.18}$$

The section modulus will be carried-out from (9.12):

$$W_t = \frac{\pi R^3}{2} = \frac{\pi D^3}{16} \,. \tag{9.19}$$

If the circular cross-section is hollow of external radius R and internal radius r, then the polar moment of inertia I_t and the section modulus W_t will be obtained by the expressions:

$$I_t = 2\pi\int_r^R \rho^3 d\rho = 2\pi(\rho^4/4)\Big|_r^R = \frac{\pi}{2}\left(R^4 - r^4\right). \tag{9.20}$$

It is obtained after substitution $\alpha = \dfrac{r}{R}$:

$$I_t = \frac{\pi R^4}{2}\left(1 - \alpha^4\right); \qquad W_t = \frac{\pi R^3}{2}\left(1 - \alpha^4\right). \tag{9.21}$$

9.3.3. DESIGN OF THE CIRCULAR BEAMS

The main restriction is the biggest shearing stresses in the beam working in pure torsion conditions to be smaller than allowable ones:

$$\tau_{\max} = \frac{|T_{\max}|}{W_t} \le \tau_{allow} \,,$$

$$W_t \ge \frac{|T_{\max}|}{\tau_{allow}} \,.$$

The diameter of the beam will be carried-out applying the condition mentioned above:

- solid circular cross-section

$$D \ge \sqrt[3]{\frac{16|T_{\max}|}{\pi\tau_{allow}}} \,;$$

- hollow circular cross-section

$$D \ge \sqrt[3]{\frac{16|T_{\max}|}{\pi(1-\alpha^4)\tau_{allow}}} \,.$$

9.4. BEAMS OF SOLID NON-CIRCULAR CROSS-SECTION

The determination of the stresses in a beam of solid non-circular cross-section working in pure torsion conditions is a very complicated problem and it can not be solved by the Strength of materials methods. The reason is the different type of deformation leading to the Bernoulli's hypothesis invalidity. The beam sections warp and thus the shearing stresses distribution change essentially. Then, to determine the shearing strains the mutual twist as well as the warping of the beam sections has to be taken into account. The strict solution of the problem is done by the Theory of elasticity.

Some special features of the shearing stresses distribution in the non-circular cross-sections can be noted: If the cross-section of the beam has external corners, then, the shearing stresses in these corners are equal to zero; If the beam surface is free of load, then, the shearing stresses in the sections situated normally to the beam's contour are also equal to zero.

The Theory of elasticity methods gives the equations about the shearing stresses distribution in common cross-sections. In the case of more complex cross-section the shearing stresses distribution might be obtained by the analogy method of *Prandtl*[1].

The beam of rectangular cross-section working in pure torsion is well-known problem in engineering practice. Theory of elasticity shows that the maximum shearing stresses are situated in the middle of the bigger side of the rectangle. The shearing stresses distribution in the beam of rectangular cross-section is shown in fig. 9.9.

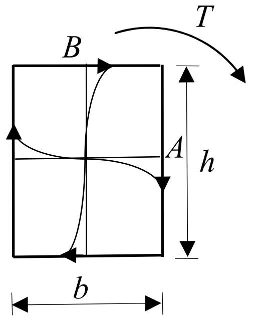


Fig. 9.9: Shearing stresses distribution

First, the ratio $n = \frac{h}{b}$ must be calculated. Then, according to this ratio, the coefficients α, β, k have to be determined using Table 9.1.

$n = \frac{h}{b}$	1	1,2	1,4	1,6	1,8	2	2,25	2,5	3
α	0,208	0,263	0,316	0,374	0,432	0,492	0,567	0,645	0,801
β	0,140	0,191	0,255	0,331	0,396	0,458	0,531	0,612	0,780
k	1,000	0,944	0,887	0,843	0,811	0,795	0,785	0,775	0,753

Table 9.1: Coefficients α, β, k depending on the ratio $n = \frac{h}{b}$

It was established, the values of coefficients α and β tend to $1/3$ when the ratio $n = \frac{h}{b}$ is very small.

[1] *Ludwig Prandtl* (1875-1953) is a German physician, professor in Hannover and Goettingen.

The polar moment of inertia and the polar section modulus will be found applying the formulas:

$$I_t = \beta b^4; \quad W_t = \alpha b^3 . \tag{9.27}$$

The shearing stresses in the typical cross-sectional points A and B are obtained by equations:

$$\tau_A = \tau_{\max} = \frac{T}{W_t}; \qquad \tau_B = k\tau_A . \tag{9.28}$$

9.5. STATE OF STRAIN

The expression (9.9) is written in a form:

$$d\varphi = \frac{T}{GI_t} dx . \tag{9.29}$$

Then, it is obtained after integration:

$$\varphi(x) = \int \frac{T}{GI_t} dx + C_1 . \tag{9.30}$$

To determine the constant C_1 the *boundary condition* must be used. If the beam has one segment, then the twist in the fixed support is equal to zero. However, if the beam has many segments, then the torsion moment function will be different for each one of them. Consequently, according to (9.30), the function of $\varphi(x)$ will be different for every segment, too. As an example, if the beam has two segments, then (9.30) will take the form:

$$\varphi_1(x) = \int \frac{T_1}{GI_{t_1}} dx + C_1 ; \tag{9.31}$$

$$\varphi_2(x) = \int \frac{T_2}{GI_{t_2}} dx + C_2 . \tag{9.32}$$

The integration constants C_1 and C_2 will be found using the boundary conditions:

- the twist in the fixed support is equal to zero;
- the twists in the boundary section are equal to each other.

Problem 9.5. Suppose that the beam in problem 9.2 has two segments of different polar moments of inertia and their ratio is $I_{t1}/I_{t2} = 1{,}2$. Build the $GI_{t1}\varphi(x)$ - diagram.

The expressions $T_1 = -12x + 14$ and $T_2 = -10$ of the torsion moments functions in the two segments, are substituted in (9.31) and (9.32), respectively. It is obtained:

$$GI_{t1}\,\varphi_1(x) = \int (12x + 14)dx + C_1 = -6x^2 + 14x + C_1 ;$$

$$GI_{t1}\,\varphi_2(x) = \frac{I_{t1}}{I_{t2}} \int (-10)dx + C_2 = 1{,}2(-10x) + C_2 .$$

The twist in the fixed support is equal to zero, i.e. $GI_{t1}\,\varphi_1(0) = 0$. It is carried-out: $GJ_{t1}\,\varphi_1(0) = -6.0^2 + 14.0 + C_1 = 0$, $C_1 = 0$. The twists in the boundary section are equal to each other, i.e. $GI_{t1}\,\varphi_1(2) = GI_{t1}\,\varphi_2(0)$. Then: $-6.2^2 + 14.2 + C_1 = -12.0 + C_2$ and $C_2 = 4\,kNm$.

The expressions of GI_{t1} - functions of twists are:

$$GI_{t1}\varphi_1(x) = -6x^2 + 14x ; \; GI_{t1}\,\varphi_2(x) = -12x + 4 .$$

The graphs are given in fig. 9.4. In accordance with relation (9.9) the function of twist $\varphi(x)$ has extremum in the beam section where the torsion moment function is equal to zero. Because of that, to draw the $GI_{t1}\,\varphi_1(x)$-diagram the values of x, namely $x = 0$; $x = 2$ and $x = 1{,}167$ have been used.

The angle of relative twist $\theta = \dfrac{d\varphi}{dx}$ of dimension rad/m is defined. θ is a measure of shearing strains when the pure torsion conditions exist. GI_c is the stiffness of a beam working in pure torsion conditions. It is obvious the bigger stiffness the lower strain.

When the shafts working in pure torsion conditions have been designed, the restriction about the strength as well as the stiffness must be satisfied. If the angle of relative twist θ is big, then, it obstructs the work of the shafts. To prevent this phenomenon *the restrictions about the relative twists* are given. After that, the check of the real relative twist must be made with respect to the boundary value of the angle. Finally, if the check is not obtained, the dimensions of the shaft must be increased.

9.6. STATICALLY INDETERMINATE BEAMS WORKING IN PURE TORSION CONDITION

All of the beams considered earlier were *statically determinate.* They are fixed at one end and only the torsion moment is unknown reaction. It was determined by the condition that the sum of the moments about the beam axis to be equal to zero. Then, the method of section has been applied, the torsion moments functions for every segment have been written and the torsion moment diagram has been drawn.

In the case of statically indeterminate beam, i.e. the beam fixed in both ends, the solution is different. Such beam is shown in fig.9.10.

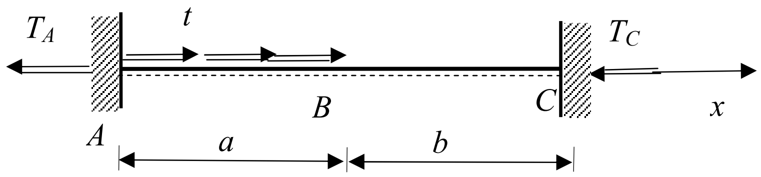


Fig. 9.10: Statically indeterminate beam working in pure torsion condition

The beam contains two unknown torsion moments as reactions while the equilibrium equation is only one. If *the number of the equilibrium equations* is subtracted from *the number of the unknowns*, then, *the degree of the statical indeterminateness* will be obtained. In our case: *two unknown torsion moments – one equilibrium equation = one time statically indeterminate problem.* The problem is *statically indeterminate* externally.

To solve the problem *the condition taking into account the type of the beam deformation* must be introduced, namely the mutual twist of the beam sections *A* and *C* must equal to zero. It can be written in the manner:

- If the segments of the beam have the same polar moment of inertia I_t:

$$\int_0^a T_1\, d\, x + \int_0^b T_2 dx = 0\,. \qquad (9.33)$$

- If the segment AB has the polar moment of inertia $I_{t,1}$, while the segment BC has $I_{t,2}$:

$$\int_0^a T_1\, d\, x + \frac{I_{t,1}}{I_{t,2}} \int_0^b T_2 dx = 0\,. \qquad (9.34)$$

Problem 9.6. Build the torsion moment diagram of the beam shown. The left segment has solid circular cross-section of diameter 0,10m, while the right segment has hollow circular cross-section of external radius 0,05m and the ratio between internal and external radius is $\alpha = \dfrac{r}{R} = 0{,}8$.

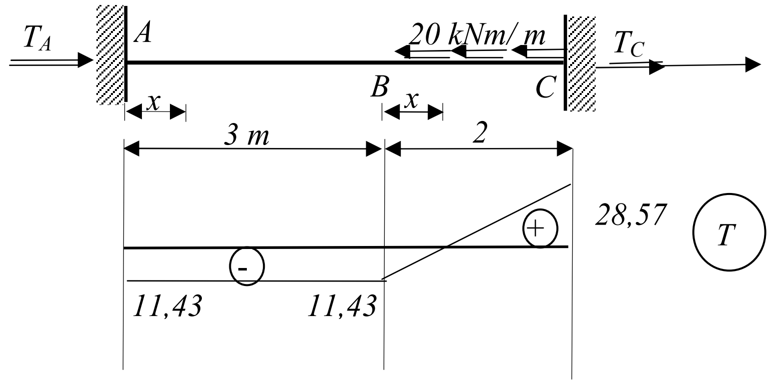


Fig. 9.10: Statically indeterminate beam

First, the moment equilibrium equation about the beam axis is written:
$\sum M_x = 0;\ \mathrm{T_A} + \mathrm{T_C} - 20.4 = 0$.

According to (9.17) the polar moment of inertia is calculated: $I_{t1} = \dfrac{\pi .0{,}05^4}{2} = 0{,}4909.10^{-5} m^4$. Further, in accordance with (9.21) the polar moment of inertia of the second segment is determined: $I_{t2} = \dfrac{\pi 0{,}05^4}{2}\left(1 - 0{,}08^4\right) = 0{,}9817.10^{-5} m^4$.

Then, the torsion moments functions of the two segments are obtained applying the method of section:

$T_1 = -T_A$; $T_2 = -T_A + 20x$.

After that, the condition (9.34) is used: $\int_0^a (-T_A)\,dx + \dfrac{I_{t,1}}{I_{t,2}}\int_0^b (-T_A + 20x)dx = 0$. The value of the unknown torsion moments is obtained after integration: $T_A = 11{,}43 kNm$.

The torsion moments expressions of the two segments are:

$T_1 = -11{,}43$; $T_2 = 20x - 11{,}43$.

Finally, the torsion moment diagram is built.